

**Heterogeneous Expectations,
Volatility and Welfare**

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Working Paper No.1
June 2000

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ABSTRACT

This paper explores the extent to which the lack of rationality of economic agents has affected the economic fluctuations of the U.S. hog market. The dynamic model of this paper adopts the framework of conventional rational expectations models and nests heterogeneity in expectations in the framework. In particular, the model assumes two types of economic agents. One (rational agent) has rational expectations and the other (boundedly rational agent) has static expectations. A log-likelihood function is constructed based on the model and the fraction of boundedly rational agents is estimated by the function. Subsequently, simulation experiments are performed to investigate the extent to which the presence of boundedly rational economic agents has affected the volatility of the economic variables of the market. In particular, two sets of artificial data are generated by the model, one set with the estimated fraction of boundedly rational agents and the other with their zero fraction. Next, the standard deviations of the quantity and price variables are computed using the simulated data and then compared. The welfare quantified as consumer surplus minus production costs is measured and compared in the same way. Empirical test results indicate that the presence of boundedly rational economic agents has increased the price and quantity volatility by 14 and 25 percent respectively, in the U.S. hog market for the period from 1945 to 1990. However, welfare turns out to be rarely affected by their presence as far as rational economic agents dominate the market.

JEL Classification: C61, C62, E32, E37

Keywords: Heterogeneous expectations, Bounded rationality, Optimal control problem, Economic fluctuations, Consumer surplus

I would like to thank Buz Brock, James Bullard, Jean-Paul Chavas, Dee Dechert, Cars Hommes, Blake LeBaron, and Thomas Lux for their helpful comments and suggestions on the previous versions of this paper. I also benefited from the comments of participants of the 1999 SCE conference at Boston College and the 2000 CeNDEF workshop at the University of Amsterdam.

I. Introduction

The purpose of this paper is to investigate the extent to which the lack of rationality (or, bounded rationality) of economic agents has affected the economic fluctuations and welfare of the U.S. hog market.

A group of articles¹ has ascribed the business cycles observed in agricultural markets to the lack of rationality of economic agents. In contrast, others, assuming the full rationality of economic agents, have ascribed them just to production lags and external shocks.² These two streams of thought are reconciled in this paper by adopting the framework of conventional rational expectations models and nesting boundedly rational economic agents in the framework.

The hog market model presented in this paper has been formulated basically following the schemes of the investment (or, inventory) decision models of Lucas and Prescott (1971), Lucas (1981), Sargent (1987, Ch. 14), West (1993) and Rosen et al (1994) among others.

In comparison with those models, however, the model of this paper assumes two different types of economic agents with regard to expectations. One type has rational expectations and the other has boundedly rational expectations. The latter is boundedly rational in the sense that this type of economic agent formulates his predictions of future prices based on past time series observations without fully understanding the market dynamics. We can find such an economic agent in a number of recent papers including Grandmont (1994), Hommes and Sorger (1998), Nerlove and Fornari (1998), Baak (1999) and Chavas (1999).

In fact, since the work of Ezekiel (1938)³, bounded rationality of economic agents has been constantly considered as causing and/or magnifying economic fluctuations. In addition, a growing number of recent papers argue that economic models incorporating boundedly rational expectations may better capture the reality of our economy⁴. However, such expectations are hardly adopted in linear quadratic dynamic models that have been well developed and widely used in various fields of modern economics.

This is partly because linear quadratic models with boundedly rational expectations can lose stationarity and generate, especially provided demand elasticity exceeds supply elasticity, exploding decision rules which obviously contradict the dynamics of any real economy. In

¹ Examples are Ezekiel (1938), Harlow (1960), Talpaz (1974), and Hayes and Schmitz (1987).

² See, for example, Jarvis (1974), Rosen et al (1994).

³ The famous cobweb theorem of Ezekiel (1938) was presented as an effort to explain hog cycles. Hog cycles, reported by economists as early as the late 19th century, are among the most volatile business cycles.

⁴ See Sargent (1993), Conlisk (1996), Brock and Hommes (1997), Arthur et al (1997), Chavas (1999), Rosser (1999) and their references.

agricultural markets, demand elasticity usually exceeds supply elasticity. However, production does not explode at all. Rosen et al (1994) and Kaufman (1996) also mention this point.

This paper nests bounded rationality in the scheme of linear quadratic dynamic models by assuming two types of economic agents instead of totally replacing rational agents with boundedly rational ones. As discussed in sections 4 and the appendix, as far as the fraction of boundedly rational agents is lower than a certain point, the model does not lose stationarity.

Based on the model presented here (the heterogeneous expectations model, hereafter), the maximum likelihood estimation (MLE, hereafter) is performed to detect the presence of boundedly rational economic agents. Then, some simulation experiments are performed to measure the extent to which their presence has affected the variances of economic variables and the welfare of the market.

The heterogeneous expectations model and its solution are presented in Section 2. Two different types of economic agents trying to maximize their linear quadratic objective functions are assumed in the framework of a dynamic equilibrium model of a competitive market.

The fraction of boundedly rational economic agents is assumed to be constant for econometric tractability. Following Baak (1999), the competitive equilibrium model is transformed into an optimal control problem with distortions. If the fraction of boundedly rational economic agents becomes zero, the distortions disappear and the model collapses to a typical optimal control problem based on the assumption of rational expectations.⁵ Therefore, the model can be regarded as a generalized version of a rational expectations model. The methodology of solving a distorted optimal control problem illustrated by McGrattan (1994) and Anderson et al (1996, AHMS hereafter) is used to solve the model.

Once the solution is obtained, the fraction of boundedly rational agents along with other deep parameters of the model can be estimated by MLE using available hog data sets. The methods of state space representations and innovations representations using the Kalman filter, illustrated by AHMS and Hansen and Sargent (1997), are used to construct the log-likelihood function. The methodologies and results of empirical tests are shown in section 3. Empirical test results indicate that the model fits with the actual hog data quite well and that a certain portion of economic agents in the U.S. hog market are boundedly rational.

After estimating the fraction of boundedly rational economic agents in section 3, simulation experiments are performed in section 4 to investigate the extent to which the presence of boundedly rational economic agents has affected the volatility and welfare of the market. In

⁵ In this case, the volatility of the market is solely ascribed to production lags and external shocks.

particular, two sets of artificial data are generated by the model, one set with the estimated fraction of boundedly rational agents and another with zero fraction of them.

First, we generate artificial data of the endogenous variables (breeding stock and hog price) using the estimated parameter values, the data for exogenous variables and the initial values of the endogenous variables. The simulated data of the endogenous variables illustrate quite similar dynamics as the actual data of the same variables. The standard deviations of the simulated price and quantity data turn out to be 84.4 and 83.6 percent of those of the actual data.

Second, another data set of the same endogenous variables is generated in the same way except for restricting the fraction of boundedly rational economic agents to be zero. That is, this data is generated in an environment in which all the economic agents are fully rational. The standard deviations of the price and quantity data simulated in this environment turn out to be 74.1 and 66.9 percent of those of the actual data. These results imply that the presence of boundedly rational economic agents has increased the price and the quantity volatility by 14 and 25 percent respectively.

The welfare quantified as consumer surplus minus production costs is also measured using the two sets of simulated data and compared in the same way. As Lucas and Prescott (1971), Dechert (1978), Lucas (1981) and Sargent (1987, Ch.14) point out, this welfare function can be regarded as a social planner's problem. If every agent is fully rational, the rational expectations equilibrium is exactly the same as the solution of the social planner's problem. In contrast to the case of volatility, welfare turns out to be rarely affected by the presence of boundedly rational agents.

What is of interest in the additional simulation experiments is that as the fraction of boundedly rational agents increases, the volatility increases and the welfare decreases more rapidly. These non-linear responses imply that if the fraction of boundedly rational agents is not constant and if the estimated fraction is the average fraction over the sample period, the volatility and the welfare loss should be greater than computed by the model of this paper. In fact, the model of this paper explains around 84 percent of the actual volatility. Issues related with this finding will be briefly discussed at the end of section 4. Then, section 5 concludes.

2. The Model

2.1 The heterogeneous expectations model in a competitive hog market

The model presented in this paper is a dynamic model of renewable resources management in which optimal decision making is incorporated with animal population dynamics. Similar models are employed in Jarvis (1974), Chavas and Klemme (1986), Rosen et al (1994), Chavas (1999) and Baak (1999). They can be regarded as investment (or, inventory) decision models applied to the agricultural sector.

Production technology

The final output of the hog industry at time t is the sum of surviving adult animals from the previous year and the one-year old animals joining the adult stock.

$$y_t = (1 - \delta)k_{t-1} + gk_{t-1} \quad (1)$$

where y_t is the final output, k_{t-1} is the adult stock at time $t - 1$, δ is the natural death rate, and g is the birth (fertility) rate per adult animal. Piglets can join the breeding stock at 8 months. Therefore, it is reasonable to assume that they become adults after one period. The death rate is assumed to be non-negative and the birth rate, positive.⁶

The final output is either marketed or held as an investment for future production (that is held for breeding). Therefore, in this industry, the amount of capital (breeding) stock at time t equals the amount of investment at time t .

$$y_t = i_t + c_t \quad (2)$$

$$k_t = i_t \quad (3)$$

where i_t is investment, c_t consumption (or sales), and k_t breeding (capital) stock at time t . Since all young males are sold (or slaughtered) at maturity, the final output held for future production is all female. Therefore, even though y_t and gk_{t-1} contains males and females, k_t contains only females.

Two types of producers

⁶ Refer to Chavas (1999) for a more detailed explanation on the production technology of hog markets.

As already mentioned, the model of this paper contains two types of producers who have different expectations for the future prices of hog. One type (a rational producer) is assumed to fully understand the market dynamics. He knows that the market price of hog is endogenously determined in the market. He purposefully acquires and uses all available information including the existence of his boundedly rational counterpart to predict future prices. This implies his prediction will be the same as that of the model.

The other type (a boundedly rational producer) is assumed to treat the hog price as an exogenous variable whose value is determined independently from market dynamics. Like in Grandmont (1994), Nerlove and Fornari (1998), and Hommes and Sorger (1998), a boundedly rational producer is assumed to formulate his hog price expectations based on time series observations. In particular, since the detrended hog price data do not fit to any AR process and look like random walks to naked eyes, a boundedly rational producer in this paper is assumed to have static expectation. That is, he holds Ezekiel (1938)'s cobweb expectation. With the exception of his approach to hog prices, however, this type is assumed to be as rational as the rational agent. The only difference between the two types of hog producers is in their predictions of future hog prices.

This paper does not investigate the reason for heterogeneity in beliefs. However, it may be worthwhile to mention that recent articles dealing with this issue usually attribute the disparity in knowledge among economic agents to differences in education (or, experience) and to deliberation (or, information) costs⁷.

If economic agents can become more rational through experience as suggested in "learning" papers such as Bray (1982) and Bray and Savin (1986), boundedly rational agents will be new entrants of the market. As far as a market has new participants to enter and experienced participants to exit, some portion of market participants can be boundedly rational.

In addition, if rationality requires implicit deliberation costs and if the relative performance of rational expectations over boundedly rational expectations is changeable, some economic agents can determine to take boundedly rational expectations as illustrated in Brock and Hommes (1997). Delong et al (1990) shows that boundedly rational agents can survive with their rational counterparts due to the additional risk they create in a market.

Objective function

⁷ See Sargent (1993), Chavas (1999), Brock and Hommes (1997), Conlisk (1996), and their references.

The integral sum of all producers in the present model is assumed to be unity without loss of generality. The fraction of boundedly rational producers is denoted by n . By definition, n is between zero and one. The goal of a producer, whether rational or boundedly rational, is assumed to be to maximize the expected present discounted value of current and future profits over an infinite horizon of time. Specifically, the objective function a producer intends to maximize is the following:

$$\begin{aligned} \max_{i_{j,t}} E_{j,0} \left\{ \sum_{t=1}^{\infty} \beta^t \left[p_t c_{j,t} - m_t c_{j,t} - h_t (k_{j,t} + \gamma_0 g k_{j,t}) - \frac{\psi_c}{2} c_{j,t}^2 - \frac{\psi_0}{2} k_{j,t}^2 \right] \right\} \\ \text{s.t.} \quad c_{j,t} = (1 - \delta) k_{j,t-1} + g k_{j,t-1} - i_{j,t} \\ k_{j,t} = i_{j,t} \end{aligned} \quad (4)$$

where p_t is the market price of an adult animal, m_t is the marketing cost of preparing an adult animal for slaughter, and h_t , $\gamma_0 h_t$ are the one-period holding costs for a breeding animal and a piglet, respectively. The quadratic term $\frac{\psi_0}{2} k_t^2$ captures the increasing costs of holding adult animals and $\frac{\psi_c}{2} c_t^2$ captures the increasing costs of preparing for slaughter. The discount factor β is assumed to be positive yet less than one. The parameters ψ_c , ψ_0 and γ_0 are assumed to be positive.

The subscript j in the endogenous state variables c , k , and the control variable i denotes that the variables are associated with a type j ($j=1$ or 2) producer. Hereafter, subscripts 1 and 2 represent rational and boundedly rational economic agents, respectively. For example, $c_{1,t}$ and $c_{2,t}$ are the quantities supplied by a rational and boundedly rational producer, respectively.

The costs $\{h_t, m_t\}_{t=0}^{\infty}$ are exogenous state variables, while the price stream $\{p_t\}_{t=0}^{\infty}$ is determined by the competitive market equilibrium. The exogenous state variables are assumed to be AR(1) processes.

$$h_t = (1 - \rho_h) h^* + \rho_h h_{t-1} + \sigma_h \varepsilon_t^h, \text{ where } h^* > 0, 0 < \rho_h < 1, 0 < \sigma_h, \varepsilon_t^h \sim \text{white noise} \quad (5)$$

$$m_t = (1 - \rho_m) m^* + \rho_m m_{t-1} + \sigma_m \varepsilon_t^m, \text{ where } m^* > 0, 0 < \rho_m < 1, 0 < \sigma_m, \varepsilon_t^m \sim \text{white noise} \quad (6)$$

Demand

The demand for hogs is assumed to be a linear function of the market price as in other agricultural articles such as Rosen et al (1994) and Chavas (1999).

$$c_t = \alpha_0 - \alpha_1 p_t + d_t, \quad \text{where } \alpha_0 > 0, \alpha_1 > 0 \quad (7)$$

$$d_t = \sigma_d \varepsilon_t^d, \quad \text{where } \varepsilon_t^d \sim \text{white noise} \quad (8)$$

It is typical to assume a linear market demand function in investment (or, inventory) decision models.⁸

Euler equation

Even though the two types of producers have the same objective function, they have different Euler equations because they predict future hog prices differently. The Euler equation of a rational producer is

$$E_t \left[\begin{aligned} &-p_t + m_t + \psi_c c_{1,t} - h_t(1 + \gamma_0 g) \\ &+ \beta(p_{t+1}(1 - \delta + g) - m_{t+1}(1 - \delta + g) - \psi_c c_{1,t+1}(1 - \delta + g)) \end{aligned} \right] = 0 \quad (9)$$

On the other hand, the Euler equation of a boundedly rational producer is

$$E_t \left[\begin{aligned} &-p_t + m_t + \psi_c c_{2,t} - h_t(1 + \gamma_0 g) \\ &+ \beta(-m_{t+1}(1 - \delta + g) - \psi_c c_{2,t+1}(1 - \delta + g)) + E_{2,t}(\beta p_{t+1}(1 - \delta + g)) \end{aligned} \right] = 0 \quad (10)$$

where $E_{2,t}$ denotes the expectations operator of a boundedly rational producer. As mentioned earlier, he has static expectations. Therefore, $E_{2,t}(p_{t+i}) = p_{t-1}$ where $i = 0, 1, 2 \dots$

The aggregate (market) Euler equation (11) is obtained by the summation of equation (9) and (10) after multiplying the fraction of rational producers, $(1 - n)$, to equation (9) and the fraction of boundedly rational producers, n , to equation (10).

$$E_t \begin{bmatrix} -(1-n)p_t + m_t + \psi_c c_{1,t} - h_t(1 + \gamma_0 g) \\ + \beta((1-n)p_{t+1}(1-\delta + g) - m_{t+1}(1-\delta + g) - \psi_c c_{1,t+1}(1-\delta + g)) \\ - np_t + n\beta(1-\delta + g)p_{t-1} \end{bmatrix} = 0 \quad (11)$$

Aggregating the constraints in the objective function (4) generate the following law of motion:

$$c_t = (1-\delta)k_{t-1} + gk_{t-1} - k_t \quad (12)$$

Then, the market as a whole is described by the aggregate Euler equation (11), the law of motion (12), the demand function (7), and the dynamic processes of the exogenous variables (5), (6) and (8).

The deep parameters of the model can be estimated with these six equations using GMM. Chavas (1999) adopts this methodology. In this paper, however, the deep parameters are estimated with the solutions of the model using MLE following the line of AHMS. Among the merits of the AHMS methodology are that it uses all the available hog data sets whether or not they are shown in the model, and that it does not require proxies for unobserved exogenous variables. In fact, three quantity data sets and one price data set are available in the U.S. hog market, and the empirical research in section 3 uses all and only these four data sets. In addition, as explained in the appendix, the methodology adopted in this paper automatically checks the stationarity condition of the model in the process of computing the solution. In the case of linear quadratic models, the stationarity of the model should be carefully checked to ensure that the model is meaningful both theoretically and empirically.

2.2 The solution of the model

A linear quadratic dynamic model with the assumptions of homogeneous rational expectations and a competitive market can usually be reformatted as an optimal control problem. Then, the model is easily solved by the well established optimal control theory. As well illustrated in Hansen and Sargent (1997), the first step of the conversion is to transform the model of a competitive economy into a social planner's problem (An HS type social planning problem, hereafter). As verified in Lucas (1981) and Sargent (1987, Ch.14), the Euler equation of a

⁸ See West (1993)

competitive market can be obtained by maximizing a social welfare criterion which equals the discounted consumer surplus minus production costs.

The model presented in section 2.1, however, cannot be represented by a social planner's problem because the model violates the first welfare theorem by assuming heterogeneity in beliefs. The economic agents are no longer homogeneous in the model. Even so, Baak(1999) shows that such a heterogeneous model can be reformatted as an optimal control problem with distortions, and therefore can be solved using computer algorithms for optimal control problems.

Baak (1999) utilizes the fact that, as shown in Becker (1985) and McGrattan (1994), a dynamic optimization problem in a competitive market with some distortions such as externalities and taxes can be replaced by a social planning problem with some side equilibrium constraints. That is, for such a model, there can be a fictional social planning problem whose first-order conditions produce the same investment sequences as the equilibrium solution of a competitive market. The social planning problem is fictional in the sense that it does not measure the precise social welfare like consumer surplus minus production cost. However, it generates the same investment sequence as a competitive market model.

In the case of the present model, a careful look at the Euler equation (11) will indicate that the equation looks like the Euler equation of a representative economic agent who faces some distortions such as externalities and taxes. The first two lines of the Euler equation are the same as in the Euler equation of a rational producer if p_t in the latter is replaced by $(1 - n)p_t$. The third line is an extra element in the Euler equation of a rational producer, and it can be treated as being imposed by an extra equilibrium condition which is associated with market distortions such as externalities and taxes. Specifically, if we include an extra term, $q_t i_t$, in the objective function and if q_t should be equal to the third line of the Euler equation (11) in the equilibrium,⁹ the aggregate Euler equation (11) can be regarded as the first-order condition of a representative agent with p_t replaced by $(1 - n)p_t$. The variable q_t is exogenous to the fictional agent but endogenous to the model.

Based on this background, the heterogeneous expectations model presented in section 2.1 can be transformed into an HS type social planning problem with side equilibrium conditions.

⁹ The fictional social planner is assumed to accept this side equilibrium condition as given, just like the demand equation. Therefore, when we solve the objective function, q_t is treated as an exogenous variable. After obtaining the first order condition of the objective function, the equilibrium condition of q_t is plugged in. This is much like the "big" K_t in the versions of Lucas and Prescott's (1971) investment model analyzed by Kydland and Prescott (1977) and Whiteman (1983).

Then the social planning problem can be reformatted as an optimal control problem with distortions. The transformation procedures and the solution are presented in the appendix.

The market equilibrium investment sequence (or, the solution of the model) will be $i_t = -F\mathbf{x}_t$, where i_t is investment at time t , and \mathbf{x}_t is the vector of state variables. Specifically \mathbf{x}_t is $[0 \ k_{t-1} \ k_{t-2} \ 1 \ h_t \ m_t \ d_t \ d_{t-1}]'$. It is explained in the appendix that the solution is unique and stationary. The equilibrium law of motion for the state variables will be $\mathbf{x}_{t+1} = A^0 \mathbf{x}_t + C\varepsilon_{t+1}$, where ε_{t+1} is a vector white noise. The specific formulae for the vector, F , and the matrices, A^0 and C , are also presented in the appendix.

The solution and the law of motion of state variables will be used in constructing the log likelihood function of the heterogeneous expectations model in section 3.1. Then the parameters of the model including the fraction of boundedly rational agents, n , will be estimated by the log likelihood function.

3. Estimations

The fraction of boundedly rational producers in the U.S. hog market is estimated in this section by MLE using observed hog data. The data sets contain annual observations for the pig crop (x_t), the number consumed (c_t), the breeding stock (k_t), and the price of an adult animal (p_t) for the U.S. during the period 1945~1990. The data are detrended by the Hodrick-Prescott filter. In each of the four data sets, the difference between the first observations in the original and in the detrended time series was added to the detrended time series to make the detrended quantity and price data take on positive values. Some information on the data is presented in Table 1.

Section 3.1 briefly explains how observed prices and quantities can be used to estimate the deep parameters of the model. The key is to construct a Gaussian log-likelihood function using the observables. First, the observables are represented by functions of state variables of the model (state space representations). Second, the state-space representations are converted into innovations representations using the Kalman filter. Third, the innovations are computed by the innovations representations. Fourth, a Gaussian likelihood function is computed by the innovations. The deep parameters can then be estimated in the way of maximizing the likelihood function.

Section 3.2 presents the estimation results.

3.1 The likelihood function

As stated above, the state variables of the heterogeneous expectations model have the following law of motion:

$$\mathbf{x}_{t+1} = A^0 \mathbf{x}_t + C \varepsilon_{t+1} \quad (13)$$

In the meantime, the four observables can be expressed as linear functions of the state variables of the model. We know from the model that $c_t = (1 - \delta + g)k_{t-1} - i_t$ and $k_t = i_t$. The function for p_t can be obtained from the function for c_t and the demand function:

$p_t = \frac{\alpha_o}{\alpha_1} - \frac{1}{\alpha_1}(1 - \delta + g)k_{t-1} + \frac{1}{\alpha_1}d_t + \frac{1}{\alpha_1}i_t$. The number of pig crops, x_t , is determined by the birth rate and the breeding stock: $x_t = gk_t$. The equilibrium solution is $i_t = -F\mathbf{x}_t$. If we add measurement errors to the observables, we can obtain the following:

$$\mathbf{y}_t = G\mathbf{x}_t + v_t \quad (14)$$

where $\mathbf{y}_t = [x_t \ k_t \ c_t \ p_t]'$, v_t is a Martingale difference sequence of measurement errors that satisfies $Ev_t v_t' = \mathbf{R}$, $E\varepsilon_{t+1} v_s' = 0$ for all $t+1 \geq s$, and

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1 - \delta + g) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-(1 - \delta + g)}{\alpha_1} & 0 & \frac{\alpha_o}{\alpha_1} & 0 & 0 & \frac{1}{\alpha_1} & 0 \end{bmatrix} + \begin{bmatrix} -gF \\ -F \\ F \\ -F/\alpha_1 \end{bmatrix}.$$

Equations (13) and (14) yield the following state space representation of the model:

$$\begin{aligned} \mathbf{x}_{t+1} &= A^0 \mathbf{x}_t + C \varepsilon_{t+1} \\ \mathbf{y}_{t+1} &= GA^0 \mathbf{x}_t + GC \varepsilon_{t+1} + v_{t+1} \end{aligned} \quad (15)$$

Then, the time-invariant innovations representation for system (15) is

$$\begin{aligned}\mathbf{x}_{t+1} &= A^0 \mathbf{x}_t + K a_t \\ \mathbf{y}_{t+1} &= G A^0 \mathbf{x}_t + a_t\end{aligned}\tag{16}$$

where $\mathbf{x}_t = \mathcal{E}[\mathbf{x}_t | \mathbf{y}_t, \mathbf{y}_{t-1}, \dots, \mathbf{y}_0, \mathbf{x}_0]$, $a_t = \mathbf{y}_{t+1} - \mathcal{E}[\mathbf{y}_{t+1} | \mathbf{y}_t, \mathbf{y}_{t-1}, \dots, \mathbf{y}_0, \mathbf{x}_0]$, K is the Kalman gain, and \mathcal{E} is the linear least squares projection operator. The innovations to \mathbf{y}_t (i.e., a_t) can be recursively computed by (16), if we assume $\mathbf{x}_0 = \mathbf{x}_0$ where \mathbf{x}_0 is the initial value of \mathbf{x}_t . That is, $a_0 = \mathbf{y}_1 - G A^0 \mathbf{x}_0$, $\mathbf{x}_1 = A^0 \mathbf{x}_0 + K a_0$, $a_1 = \mathbf{y}_2 - G A^0 \mathbf{x}_1$ and so on. Then, the log likelihood function for $\{\mathbf{y}_t\}$ is

$$L(\theta) = -\frac{1}{2} \sum_{t=0}^T \left\{ \log |\Omega_t| + \text{trace}(\Omega_t^{-1} a_t a_t') \right\} \tag{17}$$

where Ω_t is the covariance matrix of a_t . AHMS shows that Ω_t equals $G A^0 \Sigma (G A^0)' + \mathbf{R}$ where Σ is the state covariance matrix associated with the Kalman filter. The Kalman gain and the state covariance matrix can be computed by a MATLAB function *kfilter* coded by Hansen and Sargent (1997).

AHMS illustrate the analytic derivatives of the log-likelihood function and the formula for the standard errors, respectively, in the Appendix and section 11 of their paper.

3.2 The estimation results

Some parameter values are set a priori to reduce the number of parameters to be estimated. The discount factor (β) is assumed to be 0.96 following AHMS and Baak (1999). Agricultural literature usually does not contain quadratic cost terms as in Rosen et al (1994) and Chavas (1999). These terms are included in the present paper to make the objective function linear quadratic. Therefore, the coefficients of the quadratic cost terms are assumed to be close to zero: $\psi_0 = 0.0001$, $\psi_c = 0.0001$.¹⁰ Other assumptions are made in the way to increase the likelihood

¹⁰ AHMS also makes the same assumption when they estimate the model of Rosen et al (1994).

value: $\gamma_0=0.25$, $\delta=0$ ¹¹, $h^*=14$, $m^*=7$, $\sigma_h=1$, $\sigma_m=1$. It should be noted that changing the initial assumptions within reasonable boundaries does not increase the likelihood value:

$$0.1 \leq \gamma_0 \leq 2.0, 0 \leq \delta \leq 0.2, 1 \leq h^* \leq 15, 1 \leq m^* \leq 15, 0.1 \leq \sigma_h \leq 6, 0.1 \leq \sigma_m \leq 6.$$

We also assume that there is error in measuring values of quantity variables x_t , k_t and c_t . In particular, the (1,1), (2,2) and (3,3) elements of \mathbf{R} (the covariance matrix of the measurement errors) are assumed to be equal to σ_x^2 , σ_k^2 and σ_c^2 . All other elements of \mathbf{R} including the (4,4) element, σ_p^2 , are set to zero. The remaining parameters are estimated using the log-likelihood function derived from the heterogeneous expectations model of this paper.

In addition, for comparison, the parameter values are also estimated using the log-likelihood function derived from the rational expectations version of the model in which the fraction of boundedly rational economic agents, n , is set equal to zero.

In Table 2, the estimates and the standard errors of the parameters are reported. The fraction of boundedly rational ranchers, n , is significantly estimated to be 0.217. The other parameter values except for ρ_h are also significantly estimated by both models at the 5% significance level, and turn out to be within reasonable boundaries.

The specification test of the two models using log-likelihood ratio supports the heterogeneous expectations model over the rational expectations model. The null and alternative hypotheses of this test are:

$$H_0: n = 0$$

$$H_A: n \geq 0$$

In general, the specification test of two competing models (restricted and unrestricted) can be performed by computing the statistic, $2(L(\hat{\theta}) - L(\tilde{\theta}))$, since the statistic has a $\chi^2(r)$ distribution.¹² However, in the case presented in this paper, the statistic does not have a χ^2

¹¹ It may seem somewhat strange that the depreciation rate is assumed to be zero. Since the amount of capital is equal to the amount of investment in every period in this market, the zero depreciation rate does not lead unbounded accumulation of capital. The depreciation rate (natural death rate) in the U.S. hog market is known to be around 0.1. If we set $\delta = 0.1$, the log likelihood value of the estimation turns out to be a little lower than when we set $\delta = 0$. However, the estimated parameter values are affected only with negligible margins.

¹² $L(\hat{\theta})$ is the log-likelihood value of the unrestricted model (the heterogeneous expectations model) and $L(\tilde{\theta})$ is the log-likelihood value of the restricted model (the rational expectations model). The degree of

distribution because the alternative hypothesis has an inequality constraint. That is, the parameter space is restricted under the alternative hypothesis. Gouriéroux, Holly and Monfort (1982) and Andrews (1996) show that the statistic of this case is distributed as a mixture of Chi-squared distributions. Specifically, under the null hypothesis, the distribution of the statistic is $\frac{1}{2} \chi^2(0) + \frac{1}{2} \chi^2(1)$. The critical value at the 5% significance level is 2.71. The log likelihood ratio (4.915) exceeds the critical value, rejecting the null hypothesis of homogeneous full rationality.

In addition, Table 3 shows the heterogeneous expectations model produces lower mean-squared errors of one-step ahead forecasts and simulations than the model assuming homogeneous full rationality. Since the estimation results in table 2 indicate breeding stock data (k_t) contain negligible measurement errors, forecast and simulation experiments are performed with breeding stock data and price data which we assume contain no measurement errors. When σ_p^2 was estimated with other parameters, the estimated value was also negligible. This implies the assumption of no price measurement error is reasonable.

In this section the one-step ahead forecast experiments and simulation experiments were performed with two different models for their comparison. The results of these experiments along with the log-likelihood ratio test indicate that the heterogeneous expectations model better capture the dynamics of the U.S. hog market than its rational expectations version. In the following section, the extent to which the presence of boundedly rational agents has affected the volatility and the welfare of the market will be measured using the heterogeneous expectations model and its estimated parameter values.

4. Measure of Volatility and Welfare by Simulations

4.1 Volatility

Among the merits of the AHMS methodology using the Kalman filter is that it generates data for unobserved state variables (h_t, m_t, d_t) during the process of estimating the parameter values. Since now we have the data for the exogenous variables of the model and the estimated parameter values, we can do some simulation experiments. In particular, two sets of artificial data

freedom, r , is the number of restrictions. In this paper, the restriction is $n = 0$ and thus the number of restrictions is 1.

are generated by the heterogeneous expectations model, one set with the estimated fraction of boundedly rational agents and another with zero fraction of them.

First, we generate artificial data of the endogenous variables of the model (breeding stock and hog price) using the estimated parameter values, the data for exogenous variables and the initial values of the endogenous variables. Figures 1-1 and 1-2 show that the simulated data of the endogenous variables illustrate quite similar dynamics as the actual data of the same variables.

Second, another data set of the same endogenous variables is generated in the same way with the exception of restricting the fraction of boundedly rational economic agents to be zero. It should be emphasized for a clear understanding that the data are not generated by the rational expectations model introduced in the previous section. In the previous section, one set of artificial data were generated by the heterogeneous expectations model and its estimated parameter values, while another set of artificial data were generated by the rational expectations model and its estimated parameter values. In this section, two sets of artificial data are generated only by the heterogeneous expectations model. The second set is generated by the heterogeneous expectations model and its estimated parameter values except for assigning not the estimated value but zero to the fraction of boundedly rational agents. That is, these data are generated by the heterogeneous model of this paper but in a fictional environment in which all the economic agents are fully rational.¹³

Table 4 show the standard deviations of actual and simulated data. The standard deviations of the simulated price and quantity data turn out to be 84.4 and 83.6 percent of those of actual data. The standard deviations of the price and quantity data simulated in the environment of homogeneous full rationality turn out to be 74.1 and 66.9 percent of those of actual data. These results imply that the presence of boundedly rational economic agents has increased the price and the quantity volatility by 14 and 25 percent respectively. These results are not surprising at all, because it was expected that the presence of boundedly rational agents would magnify the economic fluctuations. A simple cobweb diagram of linear demand and supply curves illustrates this phenomenon well.

Figures 2-1 and 2-2 depict the dynamics of markets with rational expectations and static expectations respectively. If an external shock moves a market out of equilibrium, markets with rational expectations restore the equilibrium as soon as the external shock disappears. In contrast, in markets with static expectations, the effect of an external shock is magnified as time passes by

¹³ This environment is fictional, since the empirical tests strongly indicate some fraction of economic agents in the U.S. hog market is boundedly rational.

if the demand elasticity is greater than the supply elasticity. If the demand elasticity is smaller, the effect of an external shock fades away with a long lag.

In the present model also, the presence of boundedly rational agents plays the role to magnify the effects of external shocks. The simulation experiments indicate a substantial portion of the market fluctuations should be ascribed to the presence of boundedly rational economic agents.

4.2 Welfare

The welfare quantified as consumer surplus minus production costs is also measured using the two sets of simulated data and compared in the same way. The welfare function used for this purpose is

$$\max_{i_t} E_0 \left\{ \sum_{t=1}^{\infty} \beta^t \left[\frac{\alpha_0}{\alpha_1} c_t - \frac{1}{2\alpha_1} c_t^2 + \frac{1}{\alpha_1} c_t d_t - m_t c_t - h_t(k_t + \gamma_0 g k_t) - \frac{\psi_c}{2} c_t^2 - \frac{\psi_0}{2} k_t^2 \right] \right\} \quad (18)$$

The first part of the function, $\frac{\alpha_0}{\alpha_1} c_t - \frac{1}{2\alpha_1} c_t^2 + \frac{1}{\alpha_1} c_t d_t$, is equal to $\int_0^{c_t} \left(\frac{\alpha_0}{\alpha_1} - \frac{1}{\alpha_1} x + \frac{1}{\alpha_1} d_t \right) dx$,

that is the area under the demand curve for output associated with the capital sequence $\{k_{t+j}\}$.

The second part of the function is the production costs.

As Lucas and Prescott (1971), Dechert (1978), Lucas (1981), and Sargent (1987, Ch. 14) point out, this welfare function can be regarded as a social planner's problem. If every agent is fully rational, the rational expectations equilibrium is exactly the same as the solution of the social planner's problem. If we set n equal to zero in the model presented in section 2, the model generates the same capital (and investment) sequence as the welfare function (18).

Since the heterogeneous model of this paper is distorted by the presence of boundedly rational producers, its solution is different from the solution that maximizes the welfare function (18). Therefore, we can expect some welfare loss due to the bounded rationality of some economic agents.

Table 5 shows the welfare computed by the welfare function (18) using the data simulated by the heterogeneous expectations model. In contrast to the case of volatility, welfare was computed to be 1.862 regardless of the fraction of boundedly rational producers, implying

welfare quantified as consumer surplus minus production costs has been rarely affected by the presence of boundedly rational agents.

Table 5 also shows an interesting feature of the simulation experiments. Of interest from more simulation experiments is that as the fraction of boundedly rational agents increases, the volatility increases and the welfare decreases more rapidly. These non-linear responses imply that if the fraction of boundedly rational agents is not constant and if the estimated fraction is the average fraction over the sample period, the actual volatility and the welfare loss should be greater than computed by the model of this paper. This may explain why the simulated data capture just 84 percent of the volatility of the actual data, even with the volatility-magnifying boundedly rational agents.

Also, this finding implies that the performance of the model can be improved by permitting the fraction of boundedly rational agents to change according to the market conditions. For example, we can consider incorporating the expectations formulation function of Brock and Hommes (1997) in the present dynamic model. In the model of Brock and Hommes (1997), the portion of rational agents is a function of relative performance measure of rational expectations and information costs. Therefore, the portion is endogenously determined in the model. If we incorporate such a scheme in the present model, the model will no longer be linear and we may need a new approach to solve and estimate such a model.

5. Conclusion

In this paper, we presented a heterogeneous expectations model of the U.S. hog market and estimated the fraction of boundedly rational economic agents using the actual data of the market for the period from 1945 to 1990. The fraction of boundedly rational economic agents for the period was significantly estimated to be 22 percent. In addition, log-likelihood ratio tests, one-step ahead forecasts, and simulations supported the heterogeneous expectations model over its rational expectations version in which every agent is fully rational.

Then, we performed simulation experiments to investigate the extent to which the presence of boundedly rational economic agents has affected the volatility and welfare of the market. We generated artificial data of endogenous quantity and price variables (breeding stock and hog price) using the estimated parameter values, the data for exogenous variables and the initial values of the endogenous variables. One data set was generated in an environment where the fraction of boundedly rational economic agents is 22 percent as estimated by the actual data. On

the other hand, another data set was generated by the same model but in a different environment where all economic agents are fully rational.

Sequentially, volatility and welfare were measured using the simulated data. The standard deviations of the price and quantity data simulated by the model with 22 percent of boundedly rational agents were 1.472 and 0.601 respectively, while their counterparts computed by the same model with the exception of zero percent of boundedly rational agents were 1.293 and 0.481. These results indicate that the presence of boundedly rational agents has increased the price and quantity volatility of the U.S. hog market by 14 and 25 percent, respectively. In contrast, however, the welfare measured by consumer surplus minus production costs turned out to be rarely affected by the bounded rationality of some economic agents.

Finally, it should be noted that the heterogeneous expectations model of this paper explains only 84.4 percent of the actual price volatility and 83.6 percent of the actual quantity volatility. Further simulation experiments imply that this underestimation may be due to the constancy of the fraction of each type of economic agent. If the fraction is endogenously determined in the model as in Brock and Hommes (1997), the volatility simulated by the model can be increased. Such a model will be a generalized version of the present one, but it will require new approaches to solve and estimate such a model.

Appendix

A.1 The social planning problem with side equilibrium conditions

As stated in section 2.1, the heterogeneous expectations model of this paper can be summarized by equations (5), (6), (7), (8), (11), and (12). In this appendix, an HS type social planning problem which generates the same solution as the above equation systems is presented. The dynamic optimization problem of the fictional social planner is assumed to be the following:

$$\begin{aligned}
 & \underset{i_t}{Max} E_t \left(-\frac{1}{2} \left\{ \sum_{t=0}^{\infty} \beta^t \left[(\mathbf{s}_t - \mathbf{b}_t)' (\mathbf{s}_t - \mathbf{b}_t) + \mathbf{g}_t' \mathbf{g}_t - 2i_t q_t \right] \right\} \right) \\
 & \text{s.t. } \phi_g \mathbf{g}_t = -\phi_c c_t - \phi_i i_t + \Gamma \mathbf{k}_{t-1} + \mathbf{d}_t \\
 & \quad \mathbf{k}_t = \Delta_k \mathbf{k}_{t-1} + \Theta_k i_t \\
 & \quad \mathbf{h}_t = \Delta_h \mathbf{h}_{t-1} + \Theta_h c_t \\
 & \quad \mathbf{s}_t = \Lambda \mathbf{h}_{t-1} + \Pi c_t
 \end{aligned} \tag{A1}$$

where $\mathbf{b}_t = U_b \mathbf{z}_t$, $\mathbf{d}_t = U_d \mathbf{z}_t$, $\mathbf{z}_t = [1 \ h_t \ m_t \ d_t \ d_{t-1}]'$, $\mathbf{g}_t = [g_{1,t} \ g_{2,t}]'$, and $\mathbf{k}_{t-1} = [k_{t-1} \ k_{t-2}]'$. The vector of exogenous state variables \mathbf{z}_t is assumed to have the following transition dynamics: $\mathbf{z}_{t+1} = A_{22} \mathbf{z}_t + C_2 \varepsilon_{t+1}$, where ε_{t+1} is a vector white noise. \mathbf{k}_{t-1} is the vector of endogenous state variables. The definitions of other vectors and matrices will be presented at the end of this section.

The variable q_t should satisfy the following side constraint. As stated in section 2.1, q_t is exogenous to the fictional social planner, but endogenous to the model.

$$q_t = B_0 + B_1 k_{t-1} - B_2 k_{t-2} + B_1 d_{t-1} \tag{A2}$$

where $B_0 = \frac{\alpha_0}{\alpha_1} A_1$, $B_1 = \frac{A_1}{\alpha_1}$, $B_2 = \frac{A_1}{\alpha_1} (1 - \delta + g)$, $A_1 = n(\beta(1 - \delta + g) - 1)$

The right hand side of the constraint (A2) is the same as the third line of the Euler equation (11). To make the third line serve as a function of state and control variables, we have substituted the demand equation (7) and the law of motion (12) for p_t and c_t .

The objective function (A1) along with the constraint (A2) can replace the competitive equilibrium model in section 2.1 by defining the parameters and the matrices of the parameters such as:

$$\begin{aligned} \Lambda = \Delta_h = \Theta_h = 0; \Pi = \frac{\sqrt{1-n}}{\sqrt{\alpha_1}}; U_b = \begin{bmatrix} \frac{\alpha_0 \sqrt{1-n}}{\sqrt{\alpha_1}} & 0 & 0 & \frac{\sqrt{1-n}}{\sqrt{\alpha_1}} & 0 \end{bmatrix} \\ \phi_c = \begin{bmatrix} 1 \\ 0 \\ -f_7 \end{bmatrix} \quad \phi_i = \begin{bmatrix} 1 \\ -f_5 \\ 0 \end{bmatrix} \quad \phi_g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Gamma = \begin{bmatrix} (1-\delta+g) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad U_d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & f_6 & 0 & 0 & 0 \\ 0 & 0 & f_8 & 0 & 0 \end{bmatrix} \\ \Delta_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \Theta_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ (1-\rho_h)h^* & \rho_h & 0 & 0 & 0 \\ (1-\rho_m)m^* & 0 & \rho_m & 0 & 0 \\ (1-\rho_d)d^* & 0 & 0 & \rho_d & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_h & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_d \\ 0 & 0 & 0 \end{bmatrix} \\ f_5^2 = \psi_0, f_7^2 = \psi_c; f_5 f_6 = \gamma_0 g, f_7 f_8 = 1 \end{aligned}$$

With the definitions of the variables and the parameters illustrated above, the objective function (A1) generates the same first order condition as equation (11) with p_t replaced by the market equilibrium condition (7), the same law of motion as (12), and the same processes for exogenous state variables as equations (5), (6) and (8). Therefore, the fictional social planning problem in this appendix generates the same solution as the competitive market model with heterogeneous expectations presented in section 2.1.

In the following section, we will reformat the objective function (A1) as an optimal control problem with distortions so that we can use the optimal control theory to solve the model.

A.2 The optimal control problem

The dynamic optimization problem of a fictional social planner (A1) can be transformed into the following discounted optimal linear regulator problem:

$$Max_{u_t} E_t \left(- \sum_{t=0}^{\infty} \beta^t \left([\mathbf{x}_t \ q_t] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ q_t \end{bmatrix} + u_t' R u_t + 2 [\mathbf{x}_t \ q_t] \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} u_t \right) \right) \quad (A3)$$

$$s.t. \ \mathbf{x}_{t+1} = A_x \mathbf{x}_t + A_q q_t + B u_t + C \varepsilon_{t+1}$$

where $Q_{11} = Q$, $Q_{12} = [0000000000]'$, $Q_{21} = Q_{12}'$, $Q_{22} = 0$, $W_1 = W$, $W_2 = -\frac{1}{2}$, $A_x = A$, $A_z = 0$. The vector of state variables \mathbf{x}_t , the control variable u_t and matrices of parameters (Q, R, W, A, B, C) are defined as the following:

$$\mathbf{x}_t = [\mathbf{h}_{t-1}' \ \mathbf{k}_{t-1}' \ \mathbf{z}_t']', \ u_t = i_t,$$

$$A = \begin{bmatrix} \Delta_h & \theta_h U_c [\phi_c \ \phi_g]^{-1} \Gamma & \theta_h U_c [\phi_c \ \phi_g]^{-1} U_d \\ 0 & \Delta_k & 0 \\ 0 & 0 & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} -\theta_h U_c [\phi_c \ \phi_g]^{-1} \phi_i \\ \Theta_k \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 0 \\ C_2 \end{bmatrix}$$

$$H_s = [\Lambda \ \Pi U_c [\phi_c \ \phi_g]^{-1} \Gamma \ \Pi U_c [\phi_c \ \phi_g]^{-1} U_d - U_b] \quad H_c = [-\Pi U_c [\phi_c \ \phi_g]^{-1} \phi_i]$$

$$G_s = [0 \ U_g [\phi_c \ \phi_g]^{-1} \Gamma \ U_g [\phi_c \ \phi_g]^{-1} U_d] \quad G_c = [-U_g [\phi_c \ \phi_g]^{-1} \phi_i]$$

$$U_c = [1 \ 0 \ 0 \ 0 \ 0] \quad U_g = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R = -\frac{1}{2} (H_s' H_s + G_s' G_s) \quad Q = -\frac{1}{2} (H_c' H_c + G_c' G_c) \quad W = -\frac{1}{2} (H_c' H_s + G_c' G_s)$$

The side equilibrium condition (A2) is imposed in the following form:

$$q_t = \Theta \mathbf{x}_t + \Psi u_t \quad (A4)$$

where $\Theta = [0 \ B_1 \ -B_2 \ B_0 \ 0 \ 0 \ 0 \ B_1]$ and $\Psi = 0$.

The optimal control problem (A3) includes a discount factor (β) and cross-product terms. McGrattan(1994) and AHMS suggest converting the problem to one without cross-products and

discounting in order to simplify the analysis. Let $\tilde{\mathbf{x}}_t = \beta^{1/2} \mathbf{x}_t$, $\tilde{q}_t = \beta^{1/2} q_t$,

$$\tilde{u}_t = \beta^{1/2} u_t, \tilde{\varepsilon}_t = \beta^{1/2} \varepsilon_t, \tilde{Q}_{11} = Q_{11} - W_1 R^{-1} W_1', \tilde{Q}_{12} = Q_{12} - W_1 R^{-1} W_2',$$

$$\tilde{Q}_{22} = Q_{22} - W_2 R^{-1} W_2', \tilde{A}_x = \sqrt{\beta} (A_x - B R^{-1} W_1'), \tilde{A}_q = \sqrt{\beta} (A_q - B R^{-1} W_2'), \tilde{B} = \sqrt{\beta} B,$$

$$\tilde{\Theta} = \left(I + \Psi R^{-1} W_2' \right)^{-1} \left(\Theta - \Psi R^{-1} W_1' \right), \text{ and } \tilde{\Psi} = \left(I + \Psi R^{-1} W_2' \right)^{-1} \Psi.$$

With these definitions, we can restate the optimization problem as follows¹⁴

$$\begin{aligned} \text{Max}_{u_t} E_t \left(\sum_{t=0}^{\infty} [\tilde{\mathbf{x}}_t \quad q_t] \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_t \\ q_t \end{bmatrix} + \tilde{u}_t' R \tilde{u}_t \right) \\ \text{s.t. } \tilde{\mathbf{x}}_{t+1} = \tilde{A}_x \tilde{\mathbf{x}}_t + \tilde{A}_q \tilde{q}_t + \tilde{B} \tilde{u}_t + C \tilde{\varepsilon}_{t+1} \end{aligned} \quad (\text{A5})$$

Let $\tilde{\mathcal{A}} = \tilde{A}_x + \tilde{A}_q \tilde{\Theta}$, $\tilde{\mathcal{Q}} = \tilde{Q}_{11} + \tilde{Q}_{12} \tilde{\Theta}$, $\tilde{\mathcal{B}} = \tilde{B} + \tilde{A}_q \tilde{\Psi}$, and $\tilde{\mathcal{A}} = \tilde{A}_x - \tilde{B} R^{-1} \tilde{\Psi}' \tilde{Q}_{12}'$.

A.3 The solution

The solution of the optimal control problem (A5) is given by $\tilde{u}_t = -\tilde{F} \tilde{\mathbf{x}}_t$, where

$$\tilde{F} = \left(R + \tilde{B}' P \tilde{\mathcal{B}} \right)^{-1} \tilde{B}' P \tilde{\mathcal{A}}, \text{ and } P \text{ satisfies}$$

$$P = \tilde{\mathcal{Q}} + \tilde{\mathcal{A}}' P \tilde{\mathcal{A}} - \tilde{\mathcal{A}}' P \tilde{\mathcal{B}} (R + \tilde{B}' P \tilde{\mathcal{B}})^{-1} \tilde{B}' P \tilde{\mathcal{A}} \quad (\text{A6})$$

The equation in P , (A6), is known as the algebraic matrix Riccati equation. Since the solution matrix, \tilde{F} , contains P , solving the optimal control problem necessarily involves solving the Riccati equation (A6). The Riccati equation was solved using Vaughan's eigenvector method.¹⁵ Vaughan's eigenvector method utilizes the eigenvectors associated with the eigenvalues of the transition matrix of the state-costate evolution equation of an optimal control problem. Specifically, the algorithm of Vaughan's method is based on the fact that half of the

¹⁴ See Appendix in McGrattan(1994), Appendix B.3 in AHMS.

¹⁵ Vaughan's eigenvector method is well explained in Hansen, McGrattan and Sargent (1994). The computer codes used to implement the algorithm of Vaughan's eigenvector method for an economy distorted by heterogeneous agents are available from the author by request.

eigenvalues of the transition matrix are less than unity while the other half are greater than unity as far as the optimal control problem has a unique stationary solution. Therefore, if the computer codes generate a solution using Vaughan's method, the solution should be stationary. If the model violates any regulatory condition required for stationarity, the algorithm for Vaughan's method can not be implemented. Simulation experiments in section 4 show that the heterogeneous expectations model of this paper generates a stationary solution if the fraction of boundedly rational agents is less than 55 percent.

The solution of the original problem is the following:¹⁶

$$i_t = -F\mathbf{x}_t \text{ where } F = \left(R + W_2' \Psi \right)^{-1} \left(R\tilde{F} + W_1' + W_2' \Theta \right) \quad (\text{A7})$$

Sequentially, the equilibrium law of motion for \mathbf{x}_t is $\mathbf{x}_{t+1} = A^0 \mathbf{x}_t + C\varepsilon_{t+1}$, where $A^0 = A_x + A_q \Theta - A_q \Psi F - BF = \beta^{-1/2} (\mathbf{A} - \mathbf{B}F)$.

With some simplifying assumptions ($\psi_0 = \psi_c = 0$), this solution can also be obtained by attacking the equations (5), (6), (7), (8), (11) and (12) in section 2.1 using such lag operator manipulations illustrated in Whiteman (1983) and Sargent (1987). The market equilibrium sequence of capital obtained by lag operator manipulations will have the following equation form:¹⁷

$$(1 - \eta_1 L)(1 - \eta_2 L)k_t = H_c + H_h h_t + H_m m_t + H_d d_t + H_{d1} d_{t-1} \quad (\text{A8})$$

$$\Rightarrow k_t = (\eta_1 + \eta_2)k_{t-1} - \eta_1 \eta_2 k_{t-2} + H_c + H_h h_t + H_m m_t + H_d d_t + H_{d1} d_{t-1} \quad (\text{A9})$$

$$\Rightarrow i_t = (\eta_1 + \eta_2)k_{t-1} - \eta_1 \eta_2 k_{t-2} + H_c + H_h h_t + H_m m_t + H_d d_t + H_{d1} d_{t-1} \quad (\text{A10})$$

Equation (A9) is equivalent to equation (A10) because $i_t = k_t$ in this economy as explained in section 2.1. It is obvious that $(-F)$ in equation (A7) must be the same as

$[0 \quad \eta_1 + \eta_2 \quad -\eta_1 \eta_2 \quad H_c \quad H_h \quad H_m \quad H_d \quad H_{d1}]$. The validity of the computer codes computing $(-F)$ can be determined by checking this condition.

¹⁶ See Appendix B.3 in AHMS and Appendix B in Hansen, McGrattan and Sargent (1994).

¹⁷ The procedure of the lag operator manipulations and the definition of the parameters in the following equations are illustrated in Baak (2000) in detail.

The two roots, η_1 and η_1^* , in equation (A8) are complex conjugates and smaller than unity in modulus as far as n is less than a certain point. If n becomes larger than the critical point, their modulus exceeds unity and the solution loses stationarity. At the same time, more than half of the eigenvalues of the state-costate transition matrix of the optimal control problem becomes greater than unity in modulus. As mentioned before, the simulation experiments in section 4 shows the critical point in this market is 55 percent.

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[Table 1] Sample Means and Standard Deviations

Variable	Mean	Std. Dev.
x_t	72.969	5.324
k_t	12.916	0.719
c_t	85.383	6.228
p_t	17.185	1.745

[Table 2] MLE Test Results of the Two Models

Parameters	Theoretical	Rational Expectations Model		Heterogeneous Expectations	
Estimated	Benchmarks	($n = 0$)		Model	
		Estimates	Std. Error	Estimates	Std. Error
n	$0 \leq n \leq 1$	n.a.	n.a.	0.217*	0.006
α_0	+	137.072*	7.424	130.796*	5.987
α_1	+	3.380*	0.419	2.971*	0.331
g	+	6.157*	0.076	6.172*	0.053
ρ_h	$0 < \rho_h < 1$	0.670	1.832	0.785**	0.404
ρ_m	$0 < \rho_m < 1$	0.875*	0.152	0.747*	0.098
σ_d	+	4.263*	0.638	3.659*	0.489
σ_x^2	+	18.474*	3.707	19.100*	2.869
σ_k^2	+	0.380*	0.075	0.139*	0.022
σ_c^2	+	16.918*	2.570	17.694*	2.058
Log Likelihood		-329.183		-324.268	

1) The asterisk (*) and the double asterisk (**) indicate that the corresponding parameter estimate is significantly different from zero at the 5 percent and at the 10 percent significance level respectively.

[Table 3] Mean Squared Errors

	One-step ahead forecasts		Simulations	
	k_t (breeding)	p_t (price)	k_t (breeding)	p_t (price)
R.E.M. ^{a)}	0.722	3.254	0.253	0.970
H.E.M. ^{b)}	0.532	3.157	0.159	0.706

a) R.E.M. = rational expectations model

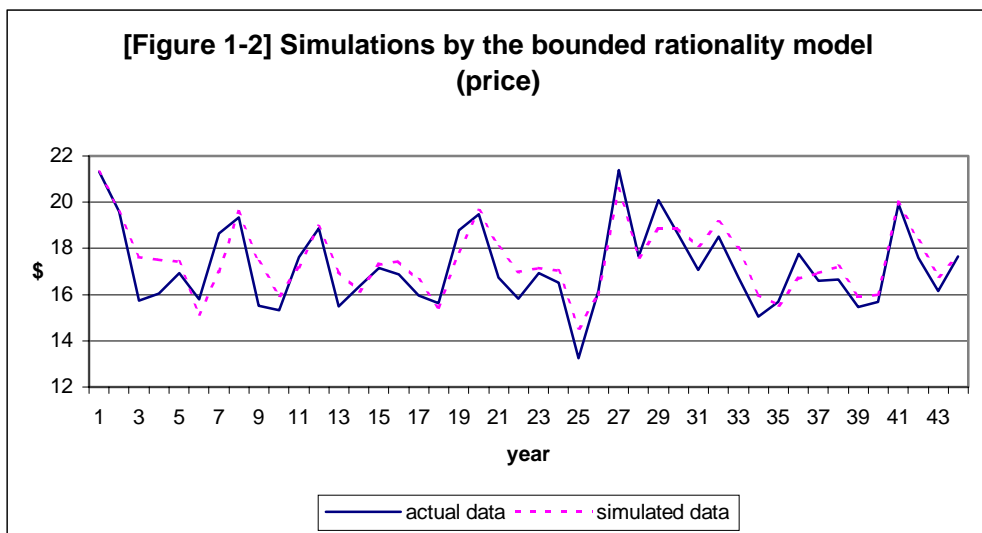
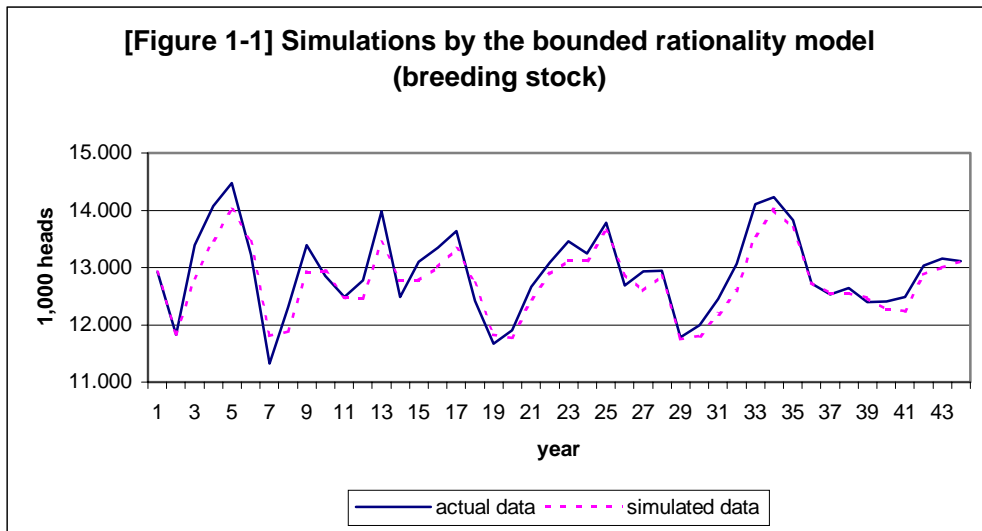
b) H.E.M. = heterogeneous expectations model

[Table 4] Standard deviations of actual and simulated data

	Actual data	Simulated data by R.E.M.	Simulated data by H.E.M.
Capital (k_t)	0.719	0.556	0.601
Price (p_t)	1.745	1.397	1.472

[Table 5] Results from simulation experiments

Fraction of boundedly rational agents	Heterogeneous Expectations Model		
	Standard deviations		Welfare
	Capital (k_t)	Price (p_t)	
0.0	0.481	1.293	1.862
0.1	0.527	1.352	1.862
0.2	0.588	1.450	1.862
0.217	0.601	1.472	1.862
0.3	0.680	1.626	1.861
0.4	0.852	2.000	1.858
0.5	1.5018	3.5369	1.836
0.55	Lose stationarity		



[Figure 2-1] The effect of an external shock: Case of rational expectations

