

Optimal Transportation Network with Discrete Entry Points in a Monocentric City*

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Abstract

The purpose of this paper is to find an optimal commuting network in a two-dimensional city. In their commuting to the business district, households choose between two modes of travel: road and rail. Road is slow but densely provided. Rail is fast, however, unlike the previous literature, it is only accessible from discrete points, namely, stations. This generates a new question: where to locate stations optimally. This paper provides a general algorithm to solve for the optimal network of a city as well as solutions under specific assumptions on the household's utility function and city shape. The paper then analyzes comparative statics of a population change.

1. Introduction

There is always a debate between the local town and the transit authority whether or not and where to construct a new railway station. Thus this paper has a clear policy motivation: when and where should we build stations? This paper not only answers this question but also provides the framework needed to analyze the effects of the resulting shocks to the city's transportation network, such as the construction of a new station and changes in population.

The research is especially relevant to the networks that are about to be built in many developing countries. The long-run economic growth that they aim for is due in large part to advances in the division of labor. However, without an extensive transportation network, markets cannot be integrated as they must be in order for the division of

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labor to advance. Therefore, many developing countries need to construct transportation networks and will benefit from the optimal design of such networks.

Furthermore, as air travel technology advances, commuting by air is becoming increasingly important in developed countries, where commuting by air is affordable. The model is readily translated into air commuting, in which case the stations are construed as airports. For simplicity, the paper uses the notation of “stations” throughout, though this term can be considered nearly interchangeable with “airports.”

There are a number of papers that deal with the spatial structure of cities, from the classic Mills-Muth model to Brueckner’s model [3], which is the standard of this class of models. However, there has not been much study on multiple transportation modes within cities. One notable exception is A. Anas and L. N. Moses [1]. Anas and Moses assume two different travel modes within a city: city streets and expressways. Their model assumes a city that has not only a dense network of city streets but also a discrete number of expressways that commuters access via city streets.

The key difference of this paper from the work of Anas and Moses is that railways are accessible to the commuters only from a certain number of stations. The additional issue then arises: where best to locate stations?

Sections 2 and 3 provide basic assumptions and notations of the model. Section 4 analyzes the household’s problem. Section 5 discusses the spatial equilibrium. Section 6 gives the optimal network in terms of the station location in the general setting. Then sections 7 and 8 present specific models: section 7 studies one-station network; section 8 studies the multi-station network. Section 9 concludes.

2. Basic Assumptions of the Model

The proposed model assumes a city with a central business district, or CBD hereafter. City population is fixed, and everyone has to commute to the CBD every day either by car or by train. There is a discrete number of railways that run from the city boundary to the CBD, with a finite number of stations along each of them. In the most general settings of the model there is no assumption about the form of the network: railways may or may not be symmetric to the CBD. The railway can be thought of alternatively as an expressway, and stations can then be entrances/exits. The railways are accessible only from a limited number of points along them, namely stations. The width of these railways is assumed to be zero. Meanwhile, city streets are accessible from anywhere in the city or, in other words, are densely provided. City streets are narrow and slow, and thus traveling on them is costlier than railway travel per unit distance. However, we will not treat congestion on city streets explicitly. At each location, a certain proportion of land is allocated to streets, with the remainder allocated to residential use. Locations in the city are expressed in polar coordinates: a radial distance from the city center and angular displacement from the nearest railway in radians.

Households are taken to be identical, and each is assumed to maximize its utility by choosing its amount of leisure time, lot size of housing, and other goods, as well as its

location in the city and its mode of travel. Time is allocated across working, commuting, and leisure. Land is owned by absentee landowners so that rent goes out of the city and the wage is the only income source for households.

3. Basic Notation of the Model

x : radial distance of a location from the center of the city

θ : angular displacement of a location in radians from the nearest radial railway

h : mode of travel: $h = 0$ is by car, $h = 1$ is by train

w : hourly wage rate (exogenous)

n : number of radial railways extending outward from the center of the city (exogenous)

H : total time endowment of each household (exogenous)

R_A : agricultural land rent per unit of land (exogenous)

N : total number of households in the city (exogenous)

$Z(x, \theta)$: consumption of other goods per household at (x, θ)

$Q(x, \theta)$: lot size per household at (x, θ)

$L(x, \theta)$: leisure time per household at (x, θ)

$R(x, \theta)$: land rent per unit of land at (x, θ)

T : lump-sum tax per household for transportation infrastructure

$G_h(x, \theta)$: generalized travel cost from (x, θ) to the city center by mode h (exogenous)

$\xi(x, \theta)$: proportion of land allocated to residential use at (x, θ) (exogenous)

4. Household Behavior

There are N identical households in this city and they are price takers. Each household is endowed with given amount of time, H . The price of a unit of other goods is normalized to unity. Since an opportunity cost of a unit amount of leisure is exogenously given as w , leisure and other goods constitute a composite commodity.

Let us construct the entire household problem so that each household chooses its location first, then its travel mode, and finally its demand for a lot, leisure, and other goods. That is, in the first stage, each household chooses its location in the city. In the second stage, it chooses its mode of travel given its location. In the third stage, it

chooses its demand for a lot, leisure, and other goods, conditional on its location and travel mode.

In practice, we solve these three steps of the households' utility-maximization problem backward. Thus we start out with the lot-leisure-other-goods choice problem. Then we proceed to the modal choice problem, which is a discrete choice problem for each household. Then at last, we proceed to its location choice problem. However, we will not treat this location-choice problem explicitly, but impose the equilibrium condition that every household achieves the same level of utility anywhere in the city.

In the third stage, each household maximizes the utility function, $U(Z, Q, L)$, conditional on its location and travel mode, given all prices. That is, it maximizes its utility as

$$\max_{Z, Q, L} U = U(Z, Q, L), \quad (4.1)$$

subject to a budget constraint that incorporates the time constraint:

$$wH - T - G_h(x, \theta) = Z + R(x, \theta)Q + wL. \quad (4.2)$$

In the budget constraint above, G_h is the generalized travel time cost, which is the sum of the money and time costs of travel, with h indexing the travel mode. Both the time cost and money costs of travel are assumed to be functions of the distance of travel. T enters as a lump sum tax, which will finance the construction cost of the railway - stations and trackage costs.¹

The Lagrangian to this maximization problem is

$$\mathcal{L} = U(Z, Q, L) + \lambda [wH - T - Z - R(x, \theta)Q - wL - G_h(x, \theta)]. \quad (4.3)$$

Solving (4.3) yields first-order conditions that have standard interpretations. Demand functions conditional on mode are obtained as the solutions to this maximization problem, where superscript $(*)$ denotes the optimal values:

$$Z_h(x, \theta) \equiv Z^*(R(x, \theta), w, T, G_h(x, \theta), H) \quad (4.4)$$

$$Q_h(x, \theta) \equiv Q^*(R(x, \theta), w, T, G_h(x, \theta), H) \quad (4.5)$$

$$L_h(x, \theta) \equiv L^*(R(x, \theta), w, T, G_h(x, \theta), H). \quad (4.6)$$

Using these gives the achievable utility level conditional on the location (x, θ) and travel mode h as the maximum value function:

$$\begin{aligned} V_h(x, \theta) &= U(Z_h(x, \theta), Q_h(x, \theta), L_h(x, \theta)) \\ &= V(R(x, \theta), w, T, G_h(x, \theta), H). \end{aligned} \quad (4.7)$$

As the second-stage problem, the household maximizes (4.7) by choosing the travel mode h conditional on location (x, θ) . For a household, maximizing $V_h(x, \theta)$ with respect to the travel mode h is the same as choosing the h that minimizes the generalized travel

¹We refer to the left-hand side of the budget constraint as the disposable income.

cost given its location, (x, θ) . We define $h^*(x, \theta)$ as the solution to this second-stage problem:

$$h^*(x, \theta) = \arg \min_h G_h(x, \theta) \quad (4.8)$$

Then we get the maximum utility level at a location (x, θ) , denoted by $V(x, \theta)$, as the following:

$$V(x, \theta) = V_{h^*}(x, \theta). \quad (4.9)$$

In the first stage, given the maximum attainable utility level at each location, each household chooses its location in the city so as to maximize the utility, $V(x, \theta)$. However, here we assume all households are identical, so that in the equilibrium they attain the same level of utility regardless of their location in the city. We define this level of utility as the global equilibrium utility level denoted by U^* . The distribution of land rent over the city adjusts to achieve this. The next section discusses this point in detail.

5. The Spatial Equilibrium

The land goes to the one who bids the most; thus the land rent $R(x, \theta)$ is the highest of all bid rents, including mode-specific commuting bid rents and the agricultural bid rent R_A . The commuting bid rent is the maximum amount that a household is willing to pay in rent for a unit area of land at each location provided that it attains the utility level of U . We denote the mode-specific bid rents by $R_h(x, \theta)$ for $h = 0, 1$:

$$R_h(x, \theta) \equiv \frac{1}{Q^c} [wH - T - G_h(x, \theta) - Z^c - wL^c] \quad (5.1)$$

where superscript (c) denotes the compensated demand for Z , Q , and L ,² given the utility level U . From above, we can see that the travel mode with lower generalized transportation cost outbids the another.

In equilibrium, land rents adjust so that attained utility levels are the same at any location in the city:

$$V(x, \theta) = U^*, \quad \forall (x, \theta) \in \mathfrak{R}, \quad (5.2)$$

where \mathfrak{R} is the entire area of the city defined as the set of locations where either (or, both) of the commuting bid rents outbids the agricultural bid rent, given the utility level set at the global equilibrium level U^* . Then the land rent is the maximum of the agricultural bid rent R_A and commuting bid rents with the utility level set at U^* :

$$R(x, \theta) \equiv \max [R_h(x, \theta), R_A], \quad h = 0, 1, \quad (5.3)$$

where $R_h(x, \theta)$ is given in (5.1). The area of the city \mathfrak{R} is then defined as the following:

$$\mathfrak{R} \equiv \{(x, \theta) \mid R(x, \theta) \geq R_A\}. \quad (5.4)$$

²These are not exactly “compensated demands” because, income (expenditure), as well as utility, is fixed instead of prices. That is, $wH - T - G_h$ is given while one of the prices R_h is to be determined endogenously.

In equilibrium, the two commuting bid rents at the boundary that separates areas of different travel modes must be equal:

$$R_0(\tilde{x}, \tilde{\theta}) = R_1(\tilde{x}, \tilde{\theta}), \quad (5.5)$$

where $(\tilde{x}, \tilde{\theta})$ is defined as any location on the boundary between areas of two modes. This implies that, at these points, the generalized travel costs are equal among these two modes:

$$G_0(\tilde{x}, \tilde{\theta}) = G_1(\tilde{x}, \tilde{\theta}). \quad (5.6)$$

At the city boundary, the higher of the two commuting bid rents must be equal to the agricultural bid rent:

$$\max \{R_0(\bar{x}, \bar{\theta}), R_1(\bar{x}, \bar{\theta})\} = R_A, \quad (5.7)$$

where $(\bar{x}, \bar{\theta})$ is any location on the boundary between the city and agricultural land. Therefore the land rent is equal to the agricultural bid rent at the city boundary:

$$R(\bar{x}, \bar{\theta}) = R_A. \quad (5.8)$$

Given the envelope theorem, the global equilibrium condition (5.2) implies:

$$\frac{\partial V(x, \theta)}{\partial x} = \frac{\partial \mathcal{L}}{\partial x} = 0 \quad (5.9)$$

$$\frac{\partial V(x, \theta)}{\partial \theta} = \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (5.10)$$

which further imply

$$\frac{\partial R(x, \theta)}{\partial x} Q^*(R(x, \theta), w, T, G(x, \theta), H) + \frac{\partial G(x, \theta)}{\partial x} = 0 \quad (5.11)$$

$$\frac{\partial R(x, \theta)}{\partial \theta} Q^*(R(x, \theta), w, T, G(x, \theta), H) + \frac{\partial G(x, \theta)}{\partial \theta} = 0, \quad (5.12)$$

where $G(x, \theta) \equiv G_{h^*}(x, \theta)$. Solving the above differential equations yields the land rent gradient as a function of the generalized transportation cost and utility level:

$$R(x, \theta) = \hat{R}(G(x, \theta), U). \quad (5.13)$$

The boundary condition (5.8) then gives the relationship between the generalized transportation cost at the city boundary and the utility level, which is monotonically decreasing:³

$$R_A = \hat{R}(G(\bar{x}, \bar{\theta}), U). \quad (5.14)$$

³Since the land rent at the city boundary is fixed at R_A , the higher the generalized transportation cost is, the lower is the utility level.

Then the lot consumption at (x, θ) in equilibrium, denoted by $Q_{h^*}(x, \theta)$, is

$$Q_{h^*}(x, \theta) = Q^*(R(x, \theta), w, T, G_{h^*}(x, \theta), H). \quad (5.15)$$

Since the city is closed, its total population is fixed:

$$N = \iint_{(x, \theta) \in \mathfrak{R}} \frac{\xi(x, \theta)}{Q_{h^*}(x, \theta)} x dx d\theta. \quad (5.16)$$

Finally, the global equilibrium utility level U^* is the utility level such that (5.16) is satisfied where the city shape \mathfrak{R} is given by (5.6) and (5.7).⁴

To solve this, we use an isomorphism. Let us define $\Phi(G)$ as the area of land where the optimal generalized transportation cost is less than or equal to G and $\phi(G) \equiv \Phi'(G)$. Since the land rent $R(x, \theta)$ is a function of the generalized transportation cost and utility level, we can define the lot consumption at locations where the optimal generalized transportation cost is G given the utility level U as $Q(G, U)$:

$$Q(G, U) \equiv Q^*(R(x, \theta), w, T, G(x, \theta), H),$$

where $R(x, \theta)$ is given in (5.13). We then get an alternative expression of (5.16) as the following:

$$\int_0^{G(\bar{x}, \bar{\theta})} \frac{\phi(\hat{G})}{Q(\hat{G}, U^*)} d\hat{G} = N. \quad (5.17)$$

This, together with (5.7), identifies the equilibrium utility level U^* and the generalized travel cost at the city boundary, denoted by \bar{G} , in terms of the exogenous variables and the generalized travel costs at all locations in the city:⁵

$$U^* = U^*(w, H, T, G_0(x, \theta), G_1(x, \theta), N, R_A) \quad (5.18)$$

$$\bar{G} = \bar{G}(w, H, T, G_0(x, \theta), G_1(x, \theta), N, R_A). \quad (5.19)$$

6. Optimizing Network

6.1. The Optimal Location of a Single Station

Let us assume that the construction cost of a railway is the sum of station and trackage costs, and that the former is exogenously given and the latter is a function of the location of the station relative to the CBD. We define s as the station cost and $K(\mathbf{x}_1)$ as the trackage cost, where $\mathbf{x}_1 \equiv (x_1, \theta_1)$ is the location of the single station in the polar coordinate, where θ_1 is set to zero without loss of generality. $K(\cdot)$ is an increasing function of the distance between \mathbf{x}_1 and the CBD.

⁴That is, we view U as the variable to control to achieve (5.16) under the constraint (5.7).

⁵These are functionals, as G_0 and G_1 are functions defined over the entire city \mathfrak{R} .

The optimal location of a station maximizes the equilibrium utility level U^* given in (5.18), and here we note explicitly that the generalized travel cost is a function of the station location, given the distance measure and the functional form of G :

$$\max_{x_1} U^* = U^*(w, H, T, G_0(x, \theta), G_1(x, \theta; x_1), N, R_A), \quad (6.1)$$

subject to the social (construction) budget constraint that tax revenue should cover the construction cost of the railway system:

$$s + K(\mathbf{x}_1) = NT. \quad (6.2)$$

However, since at the city boundary the land rent is equal to the agricultural bid rent, the maximum equilibrium utility is attained when the net income of the household at the city boundary is maximized, that is, when the generalized travel cost at the city boundary \bar{G} is minimized, given the lump-sum tax T . Thus we can rewrite the optimization problem as the following:

$$\min_{x_1} T + \bar{G}, \quad (6.3)$$

subject to the social budget (construction) constraint

$$T = \frac{s + K(\mathbf{x}_1)}{N}, \quad (6.4)$$

where \bar{G} is the generalized travel cost at the city boundary given in (5.19).

The first-order condition becomes as follows:⁶

$$\frac{1}{N} \frac{\partial K}{\partial \mathbf{x}_1} - \frac{\int_0^{\bar{G}} \frac{1}{Q(G, U)} \frac{\partial \phi(G; \mathbf{x}_1)}{\partial \mathbf{x}_1} dG}{\frac{\phi(\bar{G}; \mathbf{x}_1)}{Q(\bar{G}, U)} + \frac{\frac{\partial \hat{R}(\bar{G}, U)}{\partial G}}{\frac{\partial \hat{R}(\bar{G}, U)}{\partial U}} \int_0^{\bar{G}} \frac{\phi(G; \mathbf{x}_1)}{Q^2(G, U)} \frac{\partial Q(G, U)}{\partial U} dG} = 0, \quad (6.5)$$

where $\Phi = \Phi(G; \mathbf{x}_1)$ and $\phi = \phi(G; \mathbf{x}_1) \equiv \partial \Phi / \partial G$. The economic interpretation of the first-order condition (6.5) is that the savings from making the station closer to the CBD divided by N equals the increase in the generalized travel cost for the household at the city boundary. In (6.5), greater \bar{G} implies lower equilibrium utility, since the land rent at the boundary is equal to the agricultural land rent R_A . Furthermore, since the lot is assumed to be a normal good, lower utility means lower lot consumption at any location. Thus $\partial Q / \partial \bar{G}$ is negative, and hence the denominator of the second term is negative as well.

Solving the above first-order condition gives the optimal location of the station as the function of the set of exogenous parameters, given the distance measure and the functional form of the generalized travel cost G :

$$x_1^* = x_1(w, H, N, R_A), \quad (6.6)$$

or, since $\mathbf{x}_1 \equiv (x_1, \theta_1)$ and $\theta_1 = 0$,

$$\mathbf{x}_1^* = f_1(w, H, N, R_A). \quad (6.7)$$

⁶The problem can be simplified as

$$\min_{\mathbf{x}_1} \frac{s + K}{N} + \bar{G}$$

and the corresponding first-order condition is

$$\frac{1}{N} \frac{dK}{d\mathbf{x}_1} + \frac{d\bar{G}}{d\mathbf{x}_1} = 0,$$

while taking the total derivative of

$$\begin{aligned} N &= \int_0^{\bar{G}} \frac{\phi(G; \mathbf{x}_1)}{Q(G, U)} dG \\ R_A &= \hat{R}(\bar{G}, U) \end{aligned}$$

yields

$$\frac{d\bar{G}}{d\mathbf{x}_1} = - \frac{\int_0^{\bar{G}} \frac{1}{Q(G, U)} \frac{\partial \phi(G; \mathbf{x}_1)}{\partial \mathbf{x}_1} dG}{\frac{\phi(\bar{G}; \mathbf{x}_1)}{Q(\bar{G}, U)} + \frac{\frac{\partial \hat{R}(\bar{G}, U)}{\partial G}}{\frac{\partial \hat{R}(\bar{G}, U)}{\partial U}} \int_0^{\bar{G}} \frac{\phi(G; \mathbf{x}_1)}{Q^2(G, U)} \frac{\partial Q(G, U)}{\partial U} dG}.$$

6.2. The Optimal Location and Number of Stations

Here we study the multi-station network,⁷ where there are n stations. The locations of the n stations are represented as vectors in the polar-coordinate space: let $\mathbf{x}_1, \dots, \mathbf{x}_n$ denote those locations. One difficulty of optimizing the location of n stations arises from simultaneity; even if $n - 1$ stations are optimally located in an $(n - 1)$ -station network, adding a new station, i.e, the n th station, can make the locations of these original $n - 1$ stations suboptimal. Thus we have to decide the location of the entire set of n stations at the same time.

Another difficult issue in determining the optimal location of the n th station is whether this new station can be built on the existing trackage, or whether it is necessary to extend the existing track or to install a new line. In the most general setting, the model assumes neither that rail lines are straight, nor that they are symmetric to the CBD. The number and the shapes of the rail lines are all reflected into the track-construction-cost function K . In later sections we will have specific assumptions such as distance measure and the shape of the rail line to identify K .

We now focus on station location as the central problem. As described earlier, the optimal location of a station is contingent on the location of the other stations. The optimal location of the first station in the one-station case may well be different from that in the two-or-more stations case. Recall that in one-station case we had the optimal station location as a function of the exogenous variables:

$$\mathbf{x}_1^* = f_1(w, H, N, R_A). \quad (6.8)$$

However, now the generalized travel cost at each location is a function not only of the first station, but also of the other stations as well. Therefore, now $\Phi(G)$ is dependent on the location of other stations. Thus we have the optimal location of the first station, conditional on the locations of others, as the following:

$$\mathbf{x}_1^* = f_1(\mathbf{x}_2, \dots, \mathbf{x}_n; w, H, N, R_A); \quad (6.9)$$

and in general, for i th station,

$$\mathbf{x}_i^* = f_i(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n; w, H, N, R_A). \quad (6.10)$$

The solution will be found as a Nash equilibrium to the system of these best-response functions. This is seemingly a difficult problem; however, once we notice the indifference of having the first station in our town or the n th station in our town (if we ignore the changing-the-sign cost), it simply reduces to finding a fixed point in this system. That is, n equations that determine \mathbf{x}_1 through \mathbf{x}_n have the same functional form:

$$\begin{aligned} \mathbf{x}_1^* &= f(\mathbf{x}_2, \dots, \mathbf{x}_n; w, H, N, R_A) \\ &\vdots \\ \mathbf{x}_n^* &= f(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}; w, H, N, R_A), \end{aligned} \quad (6.11)$$

⁷The following argument is sufficiently general so that it does not assume the number of rail lines or their shapes, but only the number of stations.

where all locations \mathbf{x}_1 through \mathbf{x}_n enter interchangeably in any of the above n equations. We call this the anonymity principle. The optimal railway network is given as the solution to the system above, under the constraint that $\mathbf{x}_1 \neq \dots \neq \mathbf{x}_n$.⁸

Determining the station locations determines the construction cost of the network and commuting costs at all locations in the city as well. This then yields the maximum global-equilibrium utility level, U^{**} , which is attainable in this n -station city.

Then the optimization of the number of stations is easily determined by comparing this maximum attainable utility level of each number of stations:

$$\max_n U^{**}(w, H, N, R_A; n). \quad (6.12)$$

7. Specific Solutions I: One-Station Network

7.1. Introduction

This section presents a study on one-station network where the lot size is flexible and distance measure is of the Euclidean type. For simplicity, we make a few assumptions: first, the number of stations is assumed to be one, and so is the number of railways; second, the lump-sum tax T is constant, regardless of the location of the stations;⁹ and third, the generalized travel cost functions are linear in travel distance with a constant term. The generalized commuting-cost function via train is defined as

$$G_1(x, \theta) \equiv g_{f_1} + g_{r_1}x_1 + g_a D, \quad (7.1)$$

where subscript (f) denotes the fixed term; subscript (r), the unit cost or time of radial travel; subscript (a), the unit cost or time of access travel; x_1 , the distance of the station on the radial rail line from the center of the city ($(x_1, 0)$ is therefore the location of the station on the radial rail line); and D , the distance from the station to the location (x, θ) . Regarding travel via car, there is no access travel but only radial travel since we assume that city streets are dense. Therefore, generalized travel cost for automobile travel is defined as

$$G_0(x, \theta) \equiv g_{f_0} + g_{r_0}x. \quad (7.2)$$

It is natural to assume that $g_{r_1} < g_{r_0}$. Finally, the proportion of land allocated for residential use is assumed to be constant for all locations in the city:

$$\xi(x, \theta) = \xi, \quad \forall x, \theta. \quad (7.3)$$

⁸If the solution to (6.11) such that $\mathbf{x}_1 \neq \dots \neq \mathbf{x}_n$ exists and $f(\cdot)$ is continuous over the domain of \mathbf{x}_i 's, then there is always at least one trivial solution to this system such that

$$\mathbf{x}_1 = \dots = \mathbf{x}_n.$$

This is trivial in that all stations are located at the same place.

⁹Underlying assumption is that the trackage cost exhibits strong decreasing-cost property with respect to the distance of the track, and that the trackage cost is negligible relative to the station construction cost. Since we fix the number of station to be unity, we have the lump-sum tax T , which covers the trackage and the station construction cost, fixed as well.

7.2. Cobb-Douglas Utility and Crow-Line Distance

In addition to the common assumptions presented above, we also assume the utility function to be Cobb-Douglas:

$$U(x, \theta) \equiv \alpha \ln Z(x, \theta) + \beta \ln Q(x, \theta) + \gamma \ln L(x, \theta), \quad \alpha, \beta, \gamma > 0, \quad (7.4)$$

and distance is measured as “crow-line” distance, or Euclidean distance. The demand functions conditional on location are obtained by maximizing (7.4) subject to (4.2), for each travel mode h . Among them, conditional demand for the lot is

$$Q_h(x, \theta) \equiv Q^*(R(x, \theta), w, T, G_h(x, \theta), H) \quad (7.5)$$

$$= \frac{\beta}{\alpha + \beta + \gamma} [wH - T - G_h(x, \theta)] \frac{1}{R(x, \theta)}. \quad (7.6)$$

Thus we have the equilibrium lot consumption as the following:

$$Q_{h^*}(x, \theta) \equiv Q^*(R(x, \theta), w, T, G(x, \theta), H) \quad (7.7)$$

$$= \frac{\beta}{\alpha + \beta + \gamma} [wH - T - G(x, \theta)] \frac{1}{R(x, \theta)}. \quad (7.8)$$

Substituting the above in the differential equations in (5.9) and (5.10), together with the boundary condition (5.8), yields the land rent gradient as:¹⁰

$$R(x, \theta) = \left[\frac{wH - T - G(x, \theta)}{wH - T - \bar{G}} \right]^{\frac{\alpha + \beta + \gamma}{\beta}} R_A. \quad (7.9)$$

Finally, we can obtain \bar{G} by solving

$$N = \iint_{(x, \theta) \in \Re} \xi \frac{\alpha + \beta + \gamma}{\beta} [wH - T - G(x, \theta)]^{\frac{\alpha + \gamma}{\beta}} [wH - T - \bar{G}]^{-\frac{\alpha + \beta + \gamma}{\beta}} R_A x dx d\theta, \quad (7.10)$$

where \Re is expressed as

$$\Re = \{(x, \theta) \mid R(x, \theta) \geq R_A\}, \quad (7.11)$$

or alternatively as

$$\Re = \{(x, \theta) \mid G(x, \theta) \leq \bar{G}\}. \quad (7.12)$$

To solve the above equation, we use an isomorphism. Recall that $\Phi(G)$ is the area of the land where the generalized travel cost is less than or equal to G , and $\phi(G) \equiv \Phi'(G)$.¹¹ Then we get an alternative expression of (7.10):

$$\int_0^{\bar{G}} \frac{\phi(G; x_1)}{Q(G, U^*)} dG = N. \quad (7.13)$$

¹⁰This is monotonically decreasing in $G(x, \theta) < wH$ and no other variables in (7.9) depend on x or θ . Hence, as mentioned earlier, the boundary of market areas for two travel modes depends solely on the generalized travel costs.

¹¹Hereon, we explicitly note that these are conditional on the station location x_1 , by including it in the arguments of Φ and ϕ .

To find an analytical form of $\Phi(G)$, we have to know the shape of the city. From (7.1) and (7.2), the maximum distances from the station and from the city center to the location where the generalized travel cost is less than G are denoted respectively by D and X , as the following:

$$D(G) = \begin{cases} \text{not defined} & \forall G < g_{f_1} + g_{r_1} x_1 \\ \frac{G - g_{f_1} - g_{r_1} x_1}{g_a} & \forall G \geq g_{f_1} + g_{r_1} x_1 \end{cases} \quad (7.14)$$

$$X(G) = \begin{cases} \text{not defined} & \forall G < g_{f_0} \\ \frac{G - g_{f_0}}{g_{r_0}} & \forall G \geq g_{f_0}. \end{cases} \quad (7.15)$$

The shape of the city depends not only on \bar{G} , but also on the assumption about city-street travel. Since the travel distance on city streets is crow-line distance, the exact expression of the distance from a location (x, θ) to the station is

$$D(G) \equiv [(X(G))^2 + x_1^2 - 2x_1 X(G) \cos(\theta)]^{\frac{1}{2}}. \quad (7.16)$$

Therefore, each commuting area centering at both the station and the city center is a circle with a radius D or X , respectively. We consider four qualitatively different city shapes, which we define as phases. When \bar{G} is smaller than $g_{f_1} + g_{r_1} x_1$ or g_{f_0} , D and x , respectively, are not defined. In phase 1, $\Phi(\bar{G}) = 0$, or, in other words, there is no commuting area when \bar{G} is very small. As \bar{G} becomes larger, circles emerge sequentially or simultaneously at the center of the city and at the station, and their radii increase. To determine the analytical relationship between \bar{G} and $\Phi(\bar{G})$, we make an additional assumption that $g_{f_0} = g_{f_1}$. In this case, phase 2 corresponds to the situation in which there is only one circle around the city center, and phase 3 to that with two non-intersecting circles. Eventually, these two circles become tangent to each other. This situation is depicted in Figure 1. For larger \bar{G} , these two circles intersect - phase 4 - as shown in Figure 2. After the two circles overlap, there can be some larger \bar{G} , depending on the parameters, where one circle is completely covered by another circle. However, we exclude this by further assuming $g_a = g_{r_0}$. Analytical expressions for $\Phi(G)$ and $\phi(G)$ in each phase are presented in the Appendix.

Since attained utility level in the equilibrium is the same throughout the city including the city boundary,

$$U^* = A + (\alpha + \beta + \gamma) \ln(wH - T - \bar{G}), \quad (7.17)$$

where

$$A = -(\alpha + \beta + \gamma) \ln(\alpha + \beta + \gamma) + \alpha \ln \alpha + \beta \ln \beta + \gamma \ln \gamma - \beta \ln R_A - \gamma \ln w. \quad (7.18)$$

We further assume in this subsection and the following that the construction cost is exogenously given and independent of the station location. This implies that the global-equilibrium utility is maximized when \bar{G} , the generalized commuting cost at the city boundary, is minimized. The location of the station, x_1 , is adjusted to achieve this

optimum. The first-order condition is given as in (6.5). However, since the first term is set equal to zero as the trackage cost K is independent of x_1 , (6.5) can be rewritten as follows:

$$\frac{d\bar{G}}{dx_1} = - \frac{\int_0^{\bar{G}} \frac{1}{Q(G,U)} \frac{\partial \phi(G;x_1)}{\partial x_1} dG}{\frac{\phi(\bar{G};x_1)}{Q(\bar{G},U)} + \frac{\frac{\partial \hat{R}(\bar{G},U)}{\partial \bar{G}}}{\frac{\partial \hat{R}(\bar{G},U)}{\partial U}} \int_0^{\bar{G}} \frac{\phi(G;x_1)}{Q^2(G,U)} \frac{\partial Q(G,U)}{\partial U} dG} = 0. \quad (7.19)$$

By taking the derivative of Φ given in the Appendix C.1, we get $\partial \phi(G; x_1) / \partial x_1$ in each phase up to phase 3:¹²

	$\partial \phi(G; x_1) / \partial x_1$
phase 1	0
phase 2	0
phase 3	$-2\pi \frac{g_{r1}}{g_a^2} < 0.$

(7.20)

Since U and Q are positively related and $\hat{R}(\bar{G}, U)$ is decreasing both in \bar{G} and U , the denominator of the right-hand side in (7.19) is positive. Moreover, if the city is in phase 3, or, if the two commuting areas do not intersect, then

$$\frac{\partial \phi}{\partial x_1} < 0. \quad (7.21)$$

Substituting these in (7.19) gives

$$\frac{d\bar{G}}{dx_1} > 0, \quad (7.22)$$

implying that the optimal city has two intersecting circles of both commuting areas. Now we know that the optimal x_1 occurs in phase 4. The solution can be readily obtained numerically, once the parameter values are given; however, it is difficult to obtain the closed form of x_1 and other endogenous variables.

7.3. Conclusions

The analysis above implies that the optimal city has two intersecting commuting areas. In other words, if the city is in phase 3, it is always better to have a station closer to the center of the city so that no market area is isolated from others. This is in fact a general result. Bearing in mind that once R_A is given exogenously, decreasing \bar{G} increases U , we summarize this point as the following proposition:

Proposition 7.1. *If a city is in phase 3 then $\frac{d\bar{G}}{dx_1} > 0$.*

Proof: Since the city is in phase 3, $\partial \phi(G; x_1) / \partial x_1$ is zero for all $G \in [0, g_{f1} + g_{r1}x_1]$ and strictly negative for $G \in [g_{f1} + g_{r1}x_1, \bar{G}]$. Therefore, from (7.19) we have the result. ■

¹²It is analytically difficult to get ϕ in phase 4. However, we still get a general result, which is what we need in order to proceed in the next section, a multi-station network.

Note that the above proposition holds regardless of the city's shape. This means that the result can be applied to a multi-station city. Should there be a gap between any two commuting areas, the station(s) to such commuting areas can get closer to the CBD, so as to decrease \bar{G} without increasing the trackage cost K .¹³ Therefore, a city with multiple stations should still have no isolated commuting area. That is, an optimal city is such that all parts of the city are connected to each other. We call this the continuity principle and will refer to it later.

The economic interpretation of the tradeoff involved in the choice of the station location is as follows. Moving the station closer to the center of the city saves the commuting cost of those who travel via the rail line while it increases the area of intersection and hence results in less residential area for the same radii from the station/CBD. However, the tradeoff holds only in phase 4: if the city is in phase 3, this latter disadvantage does not arise since there is no intersection. Therefore, it is more efficient to have the station closer to the city center, and eventually the train-commuting area can connect to the city-street (direct) commuting area.

In the above two cases in particular, the optimal station location is characterized by phase 4. As mentioned above, in phase 4, there are both costs and benefits of moving the station closer to the CBD. However, at the very beginning of phase 4, that is, at the boundary between phases 3 and 4, this cost is infinitesimally small, owing to the shape of the city, whereas the benefit is finitely and strictly positive. Therefore, in these cases the optimum is inside of phase 4, in which the two commuting areas are connected to each other.

Adding another station will in general make the location of the old station suboptimal, and hence there arises a need for relocating the old stations as well as situating the new station in order to maintain an optimal railway network. We investigate the properties of multi-station cities further in the next section.

8. Specific Solutions II: Multi-Station Network

8.1. Long-Narrow City

In this section, we study a multi-station network using a specified model. Here we assume the city is built on a strip of land that has negligible vertical distance, controlled to be unity, with sufficiently long horizontal distance: angular displacement variable θ is always zero in any point in this city and thus will be eliminated from the following argument. The central business district is located at the western edge of the land, and beyond it is an ocean. The land is available infinitely far in the eastern direction. There is a railway running eastward from the CBD, with n stations located along it. We name these n stations the first, second, ..., and the n th station from the west to the east, i.e., the n th station is the east-most station. The construction cost of this railway network is thus

¹³Moving the station along the rail track toward the CBD will always do the job, even if the rail track is not a straight line from the CBD.

given as follows:

$$ns + K(x_n), \quad (8.1)$$

where s is the construction cost of each station, which is exogenous and assumed to be the same for all n stations, and $K(x_n)$ is the trackage cost to construct the track from the CBD to the n th station. Here we assume that $K(\cdot)$ has the decreasing cost property to the distance of the track, that is, $K' > 0$ and $K'' < 0$, and that $|K''|$ is sufficiently large so that the marginal cost is negligible around the n th station: $K(x_n)$ is constant at k_0 for an effective range of x_n .¹⁴ We make two more simplifying assumptions. First, the proportion of land allocated for residential use is assumed to be unity for all locations in the city:

$$\xi(x) = 1, \quad \forall x. \quad (8.2)$$

The second assumption regards the transportation costs, and holds that $g_{f_0} = g_{f_1}$ and $g_{r_0} = g_a > g_{r_1}$.

As to the households, we again assume that the utility function is of Cobb-Douglas-Leontief type:¹⁵

$$U(x) \equiv \min\{\beta Q(x), AZ(x)^\alpha L(x)^\gamma\}, \quad (8.3)$$

where $\alpha, \beta, \gamma > 0$, $\alpha + \gamma = 1$, and A is a constant term. This yields the compensated demand for a lot to be identical over location:

$$Q^c(w, U) = \frac{U}{\beta}.$$

Compensated demand functions for leisure and the other goods are independent of both the location and the land rent, which we again denote by $L^c(w, U)$ and $Z^c(w, U)$ respectively:

$$L^c(w, U) = \frac{U}{A} \left(\frac{\gamma}{\alpha w} \right)^\alpha \quad (8.4)$$

$$Z^c(w, U) = \frac{U}{A} \left(\frac{\alpha w}{\gamma} \right)^\gamma. \quad (8.5)$$

Since at the city boundary the bid rent for land is equal to the agricultural land rent, the generalized commuting cost at the city boundary \bar{G} is inversely related to the utility level, given the lump-sum tax:

$$\bar{G} = wH - T - wL^c(w, U) - Z^c(w, U) - Q^c(w, U) \quad (8.6)$$

$$= wH - T - \left[\frac{w^\gamma}{A\alpha^\alpha \gamma^\gamma} + \frac{R_A}{\beta} \right] U^*. \quad (8.7)$$

¹⁴Another natural alternative is to assume the trackage cost be linear to the distance, such as kx_n . However, the station cost s and the coefficient to the trackage cost k play similar roles in this model, and therefore many results are preserved with the current setting without complicating the analysis.

¹⁵Technically speaking, this utility function retains as much flexibility as possible by setting Z and L as composite goods, while it provides the location-independent lot size.

The problem is to maximize the equilibrium utility level U^* by controlling the number and the locations of the stations, subject to the social (construction) budget constraint that tax revenue should cover the construction cost of the railway system:

$$\max_{n, x_1, \dots, x_n} U^* \quad (8.8)$$

$$s.t. \quad ns + k_0 = NT, \quad (8.9)$$

where k_0 is the fixed part of the trackage cost. Equivalently, we can minimize the sum of the lump-sum tax and generalized commuting cost at the city boundary subject to the same constraint, since \bar{G} and U^* are inversely related. This implies that the optimum should maximize the net income, or the lot consumption at the city boundary. This is confirmed by the result that the lot-size is independent of location, as maximizing the utility implies the maximization of the lot size including that at the city boundary.

At this point, the continuity principle mentioned in the previous section plays an important role. In the optimal city, residential areas are continuous throughout the city. Therefore, with the location-independent lot size, the size of the city in terms of the location of the city boundary \bar{x} is obtained as the lot consumption times the number of households, since the vertical distance of the city is controlled to be unity:

$$\bar{x} = \frac{NU^*}{\beta}. \quad (8.10)$$

Thus the problem reduces to the maximization of the city size in the equilibrium, under the social budget constraint, given a fixed population:

$$\max_{n, x_1, \dots, x_n} \bar{x} \quad (8.11)$$

$$s.t. \quad ns + k_0 = NT. \quad (8.12)$$

As mentioned before, the commuting areas have to be continuous in the optimum. Suppose the commuting areas for the stations $1, \dots, n-1$ are continuous with each other and also with the city-street (direct) commuting area around the CBD. That is, they are located close enough to each other and to the CBD that there is no gap between any two commuting areas. Then the best location for the n th station is just outside those $n-1$ stations where the commuting area of the n th-station users is tangent to that of the $n-1$ th-station users, so that the land area where the generalized commuting cost is under \bar{G} is still continuous and maximized.¹⁶ If this is the best response of the n th station, all the other $n-1$ stations should respond in the same way as the n th station, according to the anonymity principle. That is, as long as all the other commuting areas are continuous, the last one should be located just outside them. This implies that the optimal network must be such that all the commuting areas are tangent to each other,

¹⁶Note that there would not be an overlapping of the commuting areas in the optimum in this case, unlike the previous two cases. This is because the shape of the commuting area is neither circle nor lozenge, but a long narrow rectangle.

so that they are neither detached nor overlapped. In other words, the boundary of each commuting area is such that the generalized commuting cost is exactly equal to that at the city boundary, \bar{G} . This argument yields the station locations x_i in terms of \bar{G} :¹⁷

$$x_i = \frac{\bar{G} - g_{f0}}{g_{r1}} \left[1 - \left(\frac{g_{r0} - g_{r1}}{g_{r0} + g_{r1}} \right)^i \right]. \quad (8.13)$$

Once the location of the n th station is obtained, the city boundary \bar{x} can be expressed in terms of \bar{G} as well:

$$\bar{x} = \frac{\bar{G} - g_{f0}}{g_{r1}} \left[1 - \frac{g_{r0} - g_{r1}}{g_{r0}} \left(\frac{g_{r0} - g_{r1}}{g_{r0} + g_{r1}} \right)^n \right]. \quad (8.14)$$

Note that \bar{G} and \bar{x} have a positive relationship in this equation: higher generalized commuting cost at the city boundary means larger city size.

On the contrary, from the household's point of view, higher \bar{G} implies lower net income of households at the city boundary, and thus smaller lot size. Furthermore, the smaller lot size implies a smaller city, given the lot size independent of location and fixed population. Therefore, there is another relationship between \bar{G} and \bar{x} , which is negative:

$$\bar{G} = wH - \frac{ns + k_0}{N} - \left(\frac{\beta w^\gamma}{A\alpha^{\alpha\gamma\gamma}} + R_A \right) \frac{\bar{x}}{N}, \quad (8.15)$$

which is obtained by (8.7), (8.9), and (8.10)

Finally, when we solve equations (8.14) and (8.15), \bar{G} and \bar{x} are expressed in terms of the number of stations n and exogenous parameters:

$$\bar{G} = \frac{\left[wHN - ns - k_0 + \left(\frac{\beta w^\gamma}{A\alpha^{\alpha\gamma\gamma}} + R_A \right) \left\{ 1 - \frac{g_{r0} - g_{r1}}{g_{r0}} \left(\frac{g_{r0} - g_{r1}}{g_{r0} + g_{r1}} \right)^n \right\} \frac{g_{f0}}{g_{r1}} \right]}{\left[\left(\frac{\beta w^\gamma}{A\alpha^{\alpha\gamma\gamma}} + R_A \right) \left\{ \frac{1}{g_{r1}} - \frac{g_{r0} - g_{r1}}{g_{r1}g_{r0}} \left(\frac{g_{r0} - g_{r1}}{g_{r0} + g_{r1}} \right)^n \right\} + N \right]} \quad (8.16)$$

¹⁷See Figure 3. Derivation is as follows. Since x_1 is the sum of the radii of CBD-direct commuting area and the 1st-station commuting area, and at the boundary of these two commuting areas the generalized commuting cost G is at \bar{G} , x_1 is such that

$$g_{f0} + g_{r0} \left(x_1 - \frac{\bar{G} - (g_{f0} + g_{r1}x_1)}{g_{r0}} \right) = \bar{G},$$

which simplifies to

$$2(\bar{G} - g_{f0}) = (g_{r0} + g_{r1})x_1.$$

Similarly, x_2 is such that

$$(g_{r0} - g_{r1})x_1 + 2(\bar{G} - g_{f0}) = (g_{r0} + g_{r1})x_2.$$

In general,

$$(g_{r0} - g_{r1})x_i + 2(\bar{G} - g_{f0}) = (g_{r0} + g_{r1})x_{i+1}.$$

Solving this recursive equation with the initial equation for x_1 gives the result.

and

$$\bar{x} = \frac{(wHN - ns - k_0 - g_{f_0}N) \left[1 - \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right]}{g_{r_1}N + \left(\frac{\beta w^\gamma}{A\alpha^\alpha \gamma^\gamma} + R_A \right) \left[1 - \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right]}. \quad (8.17)$$

The optimal network should maximize \bar{x} by controlling the number of stations n . Though the number of stations n is an integer and thus discrete, the expression of \bar{x} (8.17) is continuous with respect to n . Therefore, we can still treat n as continuous in optimization, which will lead us close to the solution n^* . The first-order condition identifies the optimal n in \mathbb{R} implicitly as the following:

$$\begin{aligned} & s \left[1 - \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right] g_{r_1}N \\ & + (wHN - ns - k_0 - g_{f_0}N) \left[\frac{g_{r_0} - g_{r_1}}{g_{r_0}} \ln \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right) \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right] g_{r_1}N \\ & + s \left(\frac{\beta w^\gamma}{A\alpha^\alpha \gamma^\gamma} + R_A \right) \left[1 - \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right]^2 \\ & = 0. \end{aligned} \quad (8.18)$$

8.2. Comparative Statics

In studying comparative statics, it is intuitively more informative to analyze equations (8.13), (8.14), and (8.15), rather than the first-order condition that we finally obtained. By analyzing these equations, we study the impact of population growth. Taking the total derivative of (8.14) and (8.15), we have

$$d\bar{x} = \left[\frac{1}{g_{r_1}} - \frac{g_{r_0} - g_{r_1}}{g_{r_1}g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right] d\bar{G} \quad (8.19)$$

$$\begin{aligned} d\bar{G} &= \frac{1}{N^2} \left[ns + k_0 + \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} \bar{x} \right] dN \\ &\quad - \frac{1}{N} \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} d\bar{x}. \end{aligned} \quad (8.20)$$

Solving the above yields the following:

$$\frac{d\bar{G}}{dN} = \frac{ns + k_0 + \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} \bar{x}}{N^2 + N \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} \left[\frac{1}{g_{r_1}} - \frac{g_{r_0} - g_{r_1}}{g_{r_1}g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right]} \quad (8.21)$$

$$\frac{d\bar{x}}{dN} = \frac{\left[\frac{1}{g_{r_1}} - \frac{g_{r_0} - g_{r_1}}{g_{r_1}g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right] \left(ns + k_0 + \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} \bar{x} \right)}{N^2 + N \left\{ \frac{\beta}{A} \left(\frac{w}{\gamma} \right)^\gamma \left(\frac{1}{\alpha} \right)^\alpha + R_A \right\} \left[\frac{1}{g_{r_1}} - \frac{g_{r_0} - g_{r_1}}{g_{r_1}g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n \right]}. \quad (8.22)$$

As the population grows, for a fixed number of stations, it is ambiguous whether the global-utility level rises or not. This is owing to two offsetting effects: a decrease in

the lump-sum tax and a decrease in the share of land to each household. The former is obvious since the share of the network-construction cost falls as the number of people in the city rises. The latter follows from the fact that utility level is proportional to the lot size, which is the city size divided by the population size.¹⁸ However, the generalized commuting cost at the city boundary, \bar{G} , increases unambiguously. Note that as population grows, only equation (8.15) changes: it moves out and becomes less sloped (see Figure 4; where (8.14) is labeled as equation I and (8.15) as equation II). Therefore, population growth has the effect of pushing the optimal locations of stations farther out.

Figure 5 shows (8.14) and (8.15), labeled as equations I and II respectively, before and after the population growth, for two numbers of stations. A greater number of stations n implies that the slope of (8.14) is smaller and that (8.15) is lower, yet the slope does not change. With greater population, as we have seen above, while (8.14) does not change, (8.15) shifts up and becomes less sloped; furthermore, the bigger n gets, the more extreme is this shifting up. This means that as population grows, (8.15) for $n = n_0$ and $n = n_0 + 1$ not only shift up and become less sloped but also get closer to each other. This implies, as depicted as in Figure 5, that as population grows, the optimal value of n becomes larger. Indeed, this is confirmed by taking the derivative of the first-order condition (8.18):

$$\begin{aligned} & [sg_{r_1}c'_0(n) + (wHN - ns - k_0 - g_{f_0}N)g_{r_1}c'_0(n) \\ & - c_1(n)g_{r_1}s + \left(\frac{\beta w^\gamma}{A\alpha^\alpha\gamma^\gamma} + R_A\right)\frac{2sc_0(n)c'_0(n)}{N}]dn \\ & = \left[s\left(\frac{\beta w^\gamma}{A\alpha^\alpha\gamma^\gamma} + R_A\right)\frac{c_0(n)^2}{N^2} - c_1(n)g_{r_1}(wH - g_{f_0})\right]dN \end{aligned} \quad (8.23)$$

¹⁸For the global utility level to rise, it is necessary that the rate of increase in \bar{x} is larger than that of N , when population rises. However, this is ambiguous, since

$$\begin{aligned} E_{\bar{x}:N} &= \frac{d\bar{x}}{dN} \frac{N}{\bar{x}} \\ &= \frac{\frac{ns+k_0}{\bar{x}} + \left\{\frac{\beta}{A}\left(\frac{w}{\gamma}\right)^\gamma\left(\frac{1}{\alpha}\right)^\alpha + R_A\right\}}{\left[\frac{1}{g_{r_1}} - \frac{g_{r_0}-g_{r_1}}{g_{r_1}g_{r_0}}\left(\frac{g_{r_0}-g_{r_1}}{g_{r_0}+g_{r_1}}\right)^n\right]^{-1}N + \left\{\frac{\beta}{A}\left(\frac{w}{\gamma}\right)^\gamma\left(\frac{1}{\alpha}\right)^\alpha + R_A\right\}} \end{aligned}$$

which is greater than one only when

$$\frac{ns+k_0}{\bar{x}} > \left[\frac{1}{g_{r_1}} - \frac{g_{r_0}-g_{r_1}}{g_{r_1}g_{r_0}}\left(\frac{g_{r_0}-g_{r_1}}{g_{r_0}+g_{r_1}}\right)^n\right]^{-1}N,$$

or, expressed in terms of exogenous parameters, when

$$\begin{aligned} & (ns+k_0)\left(\frac{\beta w^\gamma}{A\alpha^\alpha\gamma^\gamma} + R_A\right)\left[1 - \frac{g_{r_0}-g_{r_1}}{g_{r_0}}\left(\frac{g_{r_0}-g_{r_1}}{g_{r_0}+g_{r_1}}\right)^n\right] \\ & > (wHN - g_{f_0}N - 2(ns+k_0))g_{r_1}N. \end{aligned}$$

where

$$c_0(n) \equiv 1 - \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n > 0 \quad (8.24)$$

$$c_1(n) \equiv \frac{g_{r_0} - g_{r_1}}{g_{r_0}} \ln \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right) \left(\frac{g_{r_0} - g_{r_1}}{g_{r_0} + g_{r_1}} \right)^n < 0 \quad (8.25)$$

and

$$c'_0(n) = \frac{dc_0}{dn} > 0. \quad (8.26)$$

As population grows, for a fixed number of stations, the city grows as well and thus the generalized commuting cost at the city boundary \bar{G} increases. This further implies that the current locations of n stations will become too close to the CBD as indicated as in (8.13). However, greater population increases the need for a new station, and new-station construction indeed justifies the location of original stations to some extent. From (8.14) and (8.15), we can say that as the number of stations increases, while it is uncertain whether or not the global-utility level rises, it is always true that the generalized commuting cost at the city boundary \bar{G} decreases.¹⁹ Therefore, constructing a new station makes the old stations too dispersed from each other, which works as a counter force on the optimal station location when the population grows; as \bar{G} becomes smaller, the stations have to move in, so that they are closer to the CBD than before. This is confirmed by (8.13), which gives the station location in terms of \bar{G} .

9. Conclusions

In the crow-line distance case with a single station, it is straightforward to show that as the number of stations increases from one to infinity, the city's shape will converge to that drawn in the Anas-Moses paper, where the railway (expressway) is accessible from anywhere. In such a case, the railway extends to a finite distance from the city center, and radii of consecutive and overlapping train-commuting areas decrease continuously from the city center to the city boundary.

In the previous section, we saw a result that seems counterintuitive: stations are located closer to each other in the areas further from the city center. One immediate reason for this is the assumption of identical access-travel costs over the city. In reality, rural areas tend to have bus services from remote residential areas to the stations. Another theoretically more important reason is the assumption of the absentee landlord. The effect of this assumption is left for future work.

The market area served by a station becomes smaller as the city extends: marginal benefit of constructing a new station decreases with respect to the number of stations. This implies the welfare-maximizing number of stations is uniquely determined, provided that marginal cost is monotonically increasing, or constant as assumed in the previous model. Also, the solution will be interior, i.e., the optimal number of stations will be

¹⁹See Figure 6.

strictly positive, if the marginal benefit of constructing the first station is greater than its marginal cost.

According to the continuity principle, any cluster of population residing near but discontinuously from a nearby large city is suboptimal. This implies that any subcenter that is detached from the main center is inefficient.

One of the main shocks that makes the network suboptimal is the population change. When N is growing, it is impossible to set the station locations optimally over time. However, as we have seen, constructing a new station and increases in population have offsetting effects on the optimal locations of old stations: constructing a new station makes the optimal locations of the stations move closer in while population growth has opposite effect. Whereas population growth is a continuous process over time, station construction is a one-time shock to the system. Therefore, the problem of when to build a new station can be solved analytically by studying the dynamic optimization, which involves the rate of population growth and the station construction cost as the trigger.

Another important shock to the system is the technological advances in transportation. By changing the parameters in commuting costs in the model, the effects of this shock will be analyzed straightforwardly.

Congestion externality is not explicitly treated in the model. One way to incorporate congestion may be to introduce into this two-dimensional model the bottleneck-model type of treatment of congestion developed in Arnott, de Palma, and Lindsey's [2] paper. Kraus and Yoshida [4] have extended the bottleneck model to mass-transit commuting where service is provided discretely over time. Tabuchi [5] has studied bottleneck congestion where the rail line runs parallel to a road. Integrating these models into this two-dimensional model is another possible extension. A similar externality will arise from the longer commuting time due to the increased number of stations. In this case, we have to analyze the optimal scheduling of trains, such as skip-stopping.

A. Derivation of $\Phi(G)$ and $\phi(G)$

Calculating the area of the city in phase 1 through 3 is straightforward. However, because of intersection, it is not quite so simple in phase 4. To calculate the area of the city in phase 4, it is convenient to divide the city in parts. Figure 7 shows the city in phase 4 with its area divided into four pieces, namely, Φ_1 , Φ_2 , and two Φ_3 's. Let $(\hat{x}, \hat{\theta})$ be the intersection of the boundaries of two commuting areas. Then each area is given as follows:

$$\Phi_1 = \pi \hat{x}^2 \cdot \frac{2\pi - 2\theta}{2\pi} \quad (\text{A.1})$$

$$= (\pi - \theta) \hat{x}^2 \quad (\text{A.2})$$

$$\Phi_2 = \pi \hat{D}^2 \cdot \frac{2\pi - 2\alpha}{2\pi} \quad (\text{A.3})$$

$$= (\pi - \alpha) \hat{D}^2 \quad (\text{A.4})$$

$$\Phi_3 = \frac{1}{2} x_1 \hat{x} \sin \theta \quad (\text{A.5})$$

$$= \frac{1}{2} x_1 \hat{x} [1 - \cos^2 \theta]^{\frac{1}{2}}, \quad (\text{A.6})$$

where α is the angular displacement between $(\hat{x}, \hat{\theta})$ and the CBD to the station, and \hat{D} is the distance between $(\hat{x}, \hat{\theta})$ and the station. To eliminate $\cos \theta$ from above equation, we use the following:

$$\begin{cases} x_1 = \hat{D} \cos \alpha + \hat{x} \cos \theta \\ \hat{x} = x_1 \cos \theta + \hat{D} \cos \beta \\ \hat{D} = \hat{x} \cos \beta + x_1 \cos \alpha, \end{cases} \quad (\text{A.7})$$

where $\beta \equiv \pi - \theta - \alpha$. Solving these three simultaneous equations gives

$$\cos \theta = \frac{x_1^2 + \hat{x}^2 - \hat{D}^2}{2x_1 \hat{x}}. \quad (\text{A.8})$$

Substitute this back into Φ_3 equation gives

$$\Phi_3 = \frac{1}{2} x_1 \hat{x} \left[1 - \left(\frac{x_1^2 + \hat{x}^2 - \hat{D}^2}{2x_1 \hat{x}} \right)^2 \right]^{\frac{1}{2}} \quad (\text{A.9})$$

$$= \frac{1}{4} \left[(x_1 + \hat{x} + \hat{D}) (x_1 + \hat{x} - \hat{D}) (x_1 + \hat{D} - \hat{x}) (\hat{D} + \hat{x} - x_1) \right]^{\frac{1}{2}} \quad (\text{A.10})$$

$$= \left[S (S - \hat{D}) (S - \hat{x}) (S - x_1) \right]^{\frac{1}{2}}, \quad (\text{A.11})$$

where $S \equiv \frac{1}{2} (\hat{x} + x_1 + \hat{D})$. Thus we get $\Phi(G)$ in phase 4. Finally, $\Phi(G)$ in each phase is summarized as

	G	$\Phi(G)$
phase 1	$[0, g_{f_0}]$	0
phase 2	$[g_{f_0}, g_{f_1} + g_{r_1} x_1]$	$\pi(x(G))^2$
phase 3	$\left[g_{f_1} + g_{r_1} x_1, \frac{(g_{r_1} + g_{r_0})x_1 + (g_{f_1} + g_{f_0})}{2} \right]$	$\pi(x(G))^2 + \pi(D(G))^2$
phase 4	$\left[\frac{(g_{r_1} + g_{r_0})x_1 + (g_{f_1} + g_{f_0})}{2}, \infty \right]$	$(\pi - \theta)(x(G))^2 + (\pi - \alpha)(D(G))^2$ $+ 2[S(S - x(G))(S - x_1)(S - D(G))]^{\frac{1}{2}}$

(A.12)

where

$$\theta \equiv \arccos \frac{1}{2x_1 x(G)} (x_1^2 + (x(G))^2 - (D(G))^2) \quad (\text{A.13})$$

$$\alpha \equiv \arccos \frac{1}{2x_1 D(G)} (x_1^2 + (D(G))^2 - (x(G))^2). \quad (\text{A.14})$$

Now we can solve for $\phi(G)$. First, we find various derivatives with respect to G :

$$\frac{dx(G)}{dG} = \frac{1}{g_{r_0}} \quad (\text{A.15})$$

$$\frac{dD(G)}{dG} = \frac{1}{g_a} \quad (\text{A.16})$$

$$\frac{d\theta}{dG} = \frac{\frac{D(G)}{g_a} + \frac{x_1^2 - (D(G))^2 - (x(G))^2}{2x g_{r_0}}}{2[S(S - x(G))(S - x_1)(S - D(G))]^{\frac{1}{2}}} \quad (\text{A.17})$$

$$\frac{d\alpha}{dG} = \frac{\frac{x(G)}{g_{r_0}} + \frac{x_1^2 - (D(G))^2 - (x(G))^2}{2D g_a}}{2[S(S - x(G))(S - x_1)(S - D(G))]^{\frac{1}{2}}} \quad (\text{A.18})$$

$$\frac{dS}{dG} = \frac{1}{2} \left(\frac{1}{g_{r_0}} + \frac{1}{g_a} \right). \quad (\text{A.19})$$

By using this, we get $\phi(G)$ as the following:

	$\phi(G)$
phase 1	0
phase 2	$2\pi x(G) \frac{dx(G)}{dG}$
phase 3	$2\pi x(G) \frac{dx(G)}{dG} + 2\pi D(G) \frac{dD(G)}{dG}$
phase 4	$2(\pi - \theta) x(G) \frac{dx(G)}{dG} + 2(\pi - \alpha) D(G) \frac{dD(G)}{dG} - (x(G))^2 \frac{d\theta}{dG} - (D(G))^2 \frac{d\alpha}{dG}$ $+ [S(S - x(G))(S - x_1)(S - D(G))]^{-\frac{1}{2}}$ $\cdot \left[\frac{dS}{dG} (S - x(G))(S - x_1)(S - D(G)) \right.$ $+ S \left(\frac{dS}{dG} - \frac{dx(G)}{dG} \right) (S - x_1)(S - D(G))$ $+ S(S - x(G)) \frac{dS}{dG} (S - D(G))$ $\left. + S(S - x_1)(S - x(G)) \left(\frac{dS}{dG} - \frac{dD(G)}{dG} \right) \right]$

(A.20)

where derivatives are as described as in (A.15) through (A.19).

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