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Wenkai Li
International University of Japan

I. A. Karimi
National University of Singapore

R. Srinivasan
National University of Singapore

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International University of Japan

Planning under Correlated and Truncated Price and Demand Uncertainties

Wenkai Li[a,b], I. A. Karimi[*,b], R. Srinivasan[b]
[a]Graduate School of International Management, International University of Japan, Niigata 949-7277, Japan
[b]Department of chemical and Biomolecular Engineering, National University of Singapore, 4 Engineering Drive 4, Singapore 117576

ABSTRACT

This paper presents a novel approach to handle refinery planning under correlated and truncated random demand and price uncertainties. To compute the expectation of plant revenue, which is the main difficulty for a planning problem under uncertainty, a bivariate normal distribution is used to describe demand and price. Formulae for revenue calculation under correlated and truncated price and demand are derived. It is found that the correlation and truncation of price and demand have major influences on plant net profit. A plan that ignores these factors can be far from optimal. The Type 2 service level or fill rate undercorrelated and truncated random price and demand is derived and efficiently calculated in this paper. Maximum plant net profit that satisfies certain fill rate target can thus be obtained. The proposed approach can be generally applied for modeling other chemical plants under uncertainty.

Keywords: Refinery, Planning, Uncertainty, Correlation, Truncation

1. Introduction and motivation

Due to the changing market conditions, fluctuating environment and many other unobservable factors, many parameters in an industry are uncertain. These uncertain parameters, such as fluctuating demands and volatile raw material/product prices, are challenging and motivating decision makers to seek efficient managerial tools and models to cope with the difficulty and achieve their KPIs. This ubiquitous phenomenon is thoroughly exhibited in one of the most important industry in the national economy: the refining industry. The prices and

[*] Corresponding author. Email: cheiak@nus.edu.sg

demands of crude oil, gasoline and diesel oil are highly uncertain in reality arising from uncertain global and national economic situations and indeterminate factors such as outbreaks of war, strikes and diseases, etc.

Uncertainty can be categorized into different types according to different criteria. From the time horizon point of view, uncertainty can be categorized as short-term, mid-term and long-term uncertainty. Short-term uncertainty includes day-to-day or week-to-week processing variations, canceled/rushed orders, equipment failure, etc. Mid-term uncertainty addresses horizons of one to two years and incorporates some features from short-term and long-term uncertainties[1]. Long-term uncertainty refers to raw material/final product unit price fluctuations, demand variations, and production rate changes occurring over longer time frames ranging from five to ten years.

From the process operation point of view, there are two types of uncertainties[2]: external uncertainties and internal uncertainties. External uncertainties originate from outside but have impacts on the process. They can be the feed rate and/or feed composition and recycle flows as well as flows of utilities, the temperature and pressure of the coupled operating units or market conditions. Internal uncertainties come from the unavailability of knowledge of the process. For a determined model structure, they are uncertain model parameters that are often regressed from a limited number of experimental data. They can be the kinetic parameters of reactions in a unit such as FCC (fluidized-bed catalytic cracker) or the transfer rate of a unit such as a CDU (crude distillation unit).

From the observability point of view, uncertain parameters can be categorized into two types[3]: unknown parameters and variable parameters. The exact values of unknown parameters are never known even though the expected values may be known. These parameters include model parameters determined from experimental studies such as the kinetic parameter of a reaction as well as unmeasured and unobservable disturbances such as the influence of wind and sunshine. Variable parameters are not known at the design stage, but can be specified or measured accurately at later operating stages. These include feed flow rates, product demands and process conditions.

1.1. Correlation and truncation between uncertain demand and price

Among all the uncertainties discussed above, demand uncertainty has the dominant impact on plant profit and customer satisfaction[4,5]. Until now, most research work[4,6] on uncertainty assumes that the demand and price are independent and the price is assumed to be a constant because of the difficulty in computing the bivariate integral originated from the correlated demand and price. Such methods are only as good as their underlying assumptions. Furthermore, underlying the research of most previous papers[4,6], the ranges of demand and price are assumed to be ($-\infty$ to $+\infty$). This assumption may also bring significant inaccuracies in revenue calculation. In this section, we show to what extent the real world demand and price are corrected and truncated.

1.1.1. Dependency between demand and price

Some researchers[7] have studied the factors that influence the inter-correlation between demand and price. In general, the main relationship between crude oil demand and price can be summarized in Figure 1. Many factors, such as war, strikes, etc., influence the demand for crude oil. There are also many factors, such as the inventory level, OPEC behavior, etc., influence the crude oil price. Crude oil demand has major influences on the price of crude oil. However, this influence is weaker reversely. In both the long-run and short-run, the demand for crude oil internationally is highly insensitive to changes in price[8]. By regressing real world demand and price data from *EIA* (the U.S. Energy Information Administration)[9], the correlation coefficient between gasoline (New York Harbor Gasoline Regular) price and its demand is 0.44 for the year 2003 to 2004. For world crude oil in 2003 and 2004, the correlation coefficient is 0.30 (see Appendix I for regressing details). These data show that the demand and price are far from independent. Some researchers[4] have considered the correlation among different products. However, no research work considers the correlation between demand and price so far. Considering the correlation between the price and demand and studying its influences on plant revenue are the main concern to be addressed in this paper.

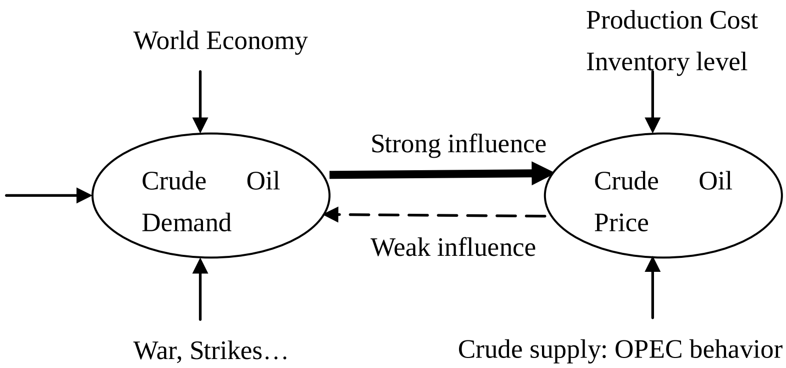

Figure 1 Dependency between crude oil demand and price

1.1.2. Truncation of demand and price

Truncation can happen when a portion of data range is not attainable on physical grounds[10]. An apparent example is that the demand or price can only take positive values. In fact, despite their random nature, most real world variables take values in a relatively narrow range. Table 1 lists some parameters estimated from the real world demand and price data (*EIA*[9]). In the second row, the mean, μ, and the standard deviation, σ, of the crude oil demand in 2003–2004 are 49.2 and 1.3 million barrels/day, respectively. The maximum and minimum demands in this period are 51.9 ($=\mu+2.1\sigma$) and 47.2 ($=\mu-1.6\sigma$) million barrels/day, respectively. Thus, we obtain the range inside which crude oil demand locates in these two years: ($\mu-1.6\sigma$, $\mu+2.1\sigma$). From the last column of Table 1, it can be seen that, real world crude oil demand and price fluctuate one to two standard deviations around their means.

Year		μ/σ	Max/Min	Range: (μ-A*σ, μ+B*σ) (A, B)
2003-2004	Total OECD crude oil demand* Million barrels/day	49.2/1.3	51.9/47.2	(1.6, 2.1)
	Crude Oil Price**	33.1/6.2	47.2/23.4	(1.6, 2.3)
1970-1999	World Total Oil Demand Million barrels/day	65.3/8.6	82.6/46.8	(2.2, 2.0)
	Crude Oil Price***	29.0/15.0	60.4/8.8	(1.3, 2.1)

*: Total OECD includes OECD Europe, Canada, Japan, South Korea, United States, and Other.

**: Brent, US$/barrel; ***: Venezuelan Tia Juana Light, US$/barrel (in 1999 real dollar)

Table 1 Ranges of the real world crude oil demand and price

Figure 2 shows the normally distributed truncated and non-truncated density functions. The difference between the truncated and non-truncated density functions is obvious. Thus, using a non-truncated density function to approximate a real world truncated density function may bring significant inaccuracies in revenue calculation and a planning strategy based on this may far from optimal.

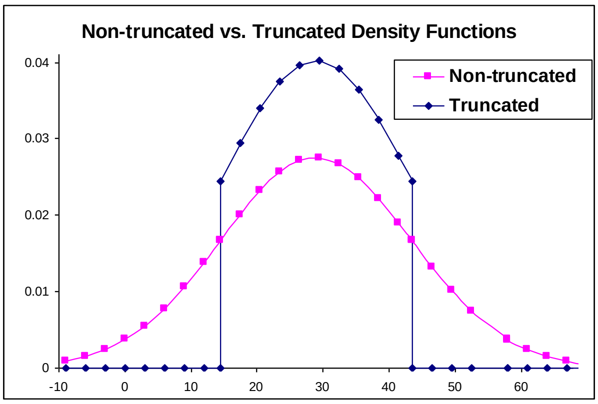

Figure 2 Normally distributed (μ=29,σ=14.5) non-truncated and truncated (truncation range: $\mu-\sigma$, $\mu+\sigma$) density functions

Thomopoulos[11] recognized that the impossibility of negative demands effectively truncates normally distributed demand patterns. He used left-truncated normal distributions to determine safety stocks and shows how the truncated normal distribution can be used to more accurately estimate the safety stock. Bookbinder and Lordahl[12] also used the left-truncated normal distribution to simulate the stochastic nature of demand patterns to set the re-order point. Johnson A. et al.[13] used univariate left-truncated normal distribution to improve the achieved service level. They realized the significant computational errors in applications led by the non-truncated normal distribution assumption. Johnson A. et al.[14] extended the univariate left truncated normal distribution to BLTN (bivariate left truncated normal distribution) and derived formula to approximate the cumulative distribution function of BLTN. They then implemented the derived formula into EXCEL. However, there exists an error in their extension which will be discussed later.

1.2. Solution approaches

Stochastic programming deals with problems in which some parameters incorporated into the objective or constraints are uncertain. These uncertain parameters are usually described by probability distributions or by possible scenarios in stochastic programming. Stochastic programming mainly consists of recourse models[15] and chance constrained programming[4,6], distinguished by the methods of describing uncertain parameters and the algorithms of solving the model.

1.2.1. Recourse model and chance constrained programming

Recourse models use corrective actions (usually penalty functions) to compensate for the violation of constraints arising during the realization of uncertainty. The two-stage model is one of the main paradigms of recourse models. Two-stage model divides the decision variables into two stages. The first-stage variables are those that have to be decided right now before future realization of uncertain parameters. Then, the second-stage variables are those used as corrective measures or as recourse against any infeasibilities arising during the realization of the uncertainty.

Because the exact values of the penalty terms are difficult to determine since they include intangible components such as loss of goodwill, the costs of off-specification products or outsourcing of production requirements in recourse models, in many cases of process operations, this penalty term is not available. This difficulty is overcome in chance constrained programming. Chance constrained programming (or the probabilistic approach) seeks to satisfy the constraints involved by a predetermined confidence level using the known probability density/cumulative distribution of random variables[16]. That is, rather than requiring constraints containing the uncertain parameters always to be satisfied or imposing penalties for infeasibilities, a probability of constraint satisfaction (usually called the confidence level) can be specified by the decision maker. This approach provides comprehensive information on economic achievements as a function of the desired confidence level.

Zimmermann[17] argued that the choice of the appropriate method is context-dependent, with no single theory being sufficient to model all kinds of uncertainty. A general-purpose algorithm is unlikely to solve all problems efficiently or exactly. It might be a good strategy that one applies different approaches for different problems. For design problems which penalty terms are easy to be obtained, using two-stage method is appropriate. For problems that penalty terms are difficult to estimate, chance constrained programming is suitable. If the computation speed is the main concern, fuzzy programming maybe a good choice. In this work, we extend the approach presented in Li et al.[6] for refinery planning under uncertainty to consider correlated and truncated demand and price uncertainty. A bivariate normal distribution is used to describe demand and price. The double integral for revenue calculation is reduced to several single integrals after detailed derivation. The unintegrable standard normal cumulative distribution function in the single integrals is approximated by polynomial function.

1.2.2. Revenue calculation methods

The computation of the revenue of a plant involves uncertain variables such as market demand and product price. How to compute the expectations of uncertain functions introduced by these uncertain variables generates the main difficulty in stochastic rogramming[18] and approaches in the literature differ primarily in how these expectations are computed[19]. Several approaches have been used in the literature to compute these expectations.

Clay et al.[15] applied the certainty equivalent transformation (CET) to yield a deterministic equivalent problem. Ierapetritou et al.[20] used the Gaussian quadrature formula to approximate the expected revenue. Liu et al.[21] used Monte Carlo sampling to estimate the expectation of the objective function. Li et al.[6] categorized different revenue calculation approaches into three types which include: A) Minimizing cost. The objective is to minimize the total costs and the computation of plant revenue is avoided. B). Maximizing profit I. The revenue is calculated by the product of the market price and the amount of the product produced by the plant. In this approach, it is assumed that a product can always be absorbed by the market. C). Maximizing profit II. The revenue is calculated by the product of the market price and the market demand. In this approach, it is assumed that the amount of a product is always greater than the market demand. However, the assumptions in B) and C) are not always true in the real world. As pointed out by Petkov et al.[4] and Li et al[6], in many cases, if the market demand is less than the product amount, only part of the product can be sold; otherwise if the market demand is higher than the product amount, then only part of the demand can be satisfied. The revenue should then be calculated by:

$$\text{Revenue} = E[\sum_{c}\sum_{x} c\min(P, x)] \tag{1}$$

where c is the price, P is the production rate of the product and x is the random demand. This representation implies that Revenue is not normally distributed even though x is normal. In previous works[4,6], c is assumed to be a constant and replaced by its expected value, $\bar{c}$.

1.3 Applications of modeling with uncertainty

Since Dantzig's seminal work on uncertainty appeared[22], research on uncertainty has been attracting the attention of numerous researchers. Problems that include uncertainty mainly focus on plant design, plant planning/scheduling and supply chain management.

Researchers commonly studied the design of chemical process using two-stage (operating and design stage) approach. The investment decisions (equipment sizes) are determined at the design stage and the effect of uncertainty is considered in the second stage. Wellons and Reklaitis[23] investigated the design of multiproduct batch plants under uncertainty. They suggested a distinction between "hard" and "soft" constraints and introduced penalty terms for the latter type. Analytical expressions of the expected profit objective were developed in their paper. Rooney and Biegler[3] studied the optimal process design that incorporated two types of uncertain parameters, unknown parameters and variable parameters. An extended two-stage method was proposed in their paper. Yi et al.[24] proposed *PSW* (periodic square wave) model which can provide useful data for investment decision making in a highly uncertain business environment. Uncertainty on demand, cycle time and product quality were considered. The model was used to design a batch-storage network.

Planning is essential for plants after optimal design was made. It addresses applications such as feedstock selection and disposition, as well as overall material balance and conversion optimization[15]. Clay et al.[15] studied production planning using linear two-stage approach. Uncertain parameters were presented by finite discrete probility distribution functions. Petkov et al. [4] studied the planning of multiproduct batch plants under demand uncertainty. They converted normally distributed demand into a chance constraint programming problem. The expectation of revenue was computed for correlated product demands. Lee et al.[25] proposed a general strategy for treating an open-shop batch process planning. Discrete demand pattern was used and a hybridization of the Monte Carlo simulation and simulated annealing techniques was applied in the flexible planning algorithm. They also concluded that the open-shop mode is preferred in a batch process when the inventory cost is large. Refineries are vital components of national economies. However, the study on refinery planning under uncertainty is still far from mature up to now. Clay et al.[15] used refinery planning in their case studies to illustrate their solution algorithm. Neiro et al.[26] performed supply chain optimization of refineries with the consideration of uncertainty using a scenario-based approach. Li et al.[6] studied refinery planning under uncertainty using chance constraint programming. They used approximated standard loss function to calculate the plant revenue. They also implemented Type 2 service level (fill rate) target into the planning model. For linear planning problem, the model can be solved efficiently.

In the past few years, scheduling under uncertainty is attracting the attention of more and more researchers. Balasubramanian et al.[27] studied the problem of multi-period batch plant scheduling under demand uncertainty. Scenario tree was used to describe all possible solutions of the scheduling model. They proposed a shrinking-horizon approach to approximate the multistage stochastic MILP model. By solving a series of two-stage models and implementing the decisions period by period, the computational difficulties associated with large-scale multistage model was overcome. Janak et al.[28] proposed a robust optimization methodology based on a min-max framework. Uncertainty was considered in the coefficients of the objective function and the right-hand-side parameters of inequality constraints. Several known distributions were used to describe uncertain data. Bonfill et al.[29] studied scheduling under uncertain processing times. A two-stage stochastic approach was applied whose objective was to minimize a weighted sum of the expected makespan and the expected wait times.

The study of the supply chain under uncertainty is important with the ever-changing market conditions. Gupta et al.[1] studied supply chain planning using two-stage programming. Inventories were considered in the model and hard-to-specify penalty terms were used for stockout or too low inventory levels. Applequist et al.[30] introduced a new metric for evaluating supply chain design and planning risk under uncertainty. A rational balance between the return and risk can thus be obtained. Chen and Lee[31] developed a multi-objective scheduling model for a multi-echelon supply chain network with uncertain demands and product prices. Scenarios with known probabilities were used to describe random demand. Conflicting objectives, such as fair profit distribution among all participants, safe inventory levels and maximum customer service levels, were taken into account. Guillen et al.[32] proposed a two-level framework to address the design of chemical supply chains under uncertainty. The structure of the supply chain network was decided in the strategic level and sent to the lower operational and tactical level to compute the expectation of profit under uncertainty. The profit from the lower level was sent back and evaluated in the strategic level to decide whether any changes of the supply chain structure is needed. The product price was assumed to be known a priori in their work. Alternatively, Mele et al.[33] applied agent-based approach for supply chain retrofitting under uncertainty. The demand was modeled as a set of events distributed over time horizon. Uncertain processing and transport times were incorporated via normal probability functions.

2. Revenue for correlated price and demand

Assuming independent demand and the price can cause significant discrepancies in revenue calculation due to correlated price and demand for a real world plant. In this section, the formulae for plant revenue computation considering the correlation between price and demand are derived. Instead of a constant, the price, c, is assumed to conform a two-dimensional normal distribution together with the demand, x. The corresponding pdf (probability density function) is represented by:

$$\varphi(c,x)=\frac{1}{2\pi\sigma_c\sigma_x\sqrt{1-\rho^2}}e^{\frac{-1}{2(1-\rho^2)}[\frac{(c-\bar{c})^2}{\sigma_c^2}-\frac{2\rho(c-\bar{c})(x-\theta)}{\sigma_c\sigma_x}+\frac{(x-\theta)^2}{\sigma_x^2}]} \tag{2}$$

where, $\bar{c}$ is the mean of price, σ_c is the standard deviation of price. θ is the mean of demand and σ_x is the standard deviation of demand. ρ is the correlation coefficient. The normally distributed price and demand are independent if $\rho=0$**.** Normal distribution is widely used in scientific and statistical computing because it captures the essential features of variables in broad areas such as petroleum industry[10]. Furthermore, from the Central Limit Theorem, normal distribution can be used as an approximation to some other distributions and provides the foundation for other statistical procedures because the distribution of non-normal average tends to be normal. Therefore, we focus on normally distributed demand and price in this paper.

Assuming both the demand and price take values in the range (–∞, +∞) and combining eqs (1) and (2), the revenue is $\int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} xc\varphi(c,x)dxdc$ if $x \le P$ and $\int_{c=-\infty}^{+\infty}\int_{x=P}^{+\infty} Pc\varphi(c,x)dxdc$ if $x > P$.

Then, eq (1) becomes

$$\begin{aligned}\text{Revenue} &= \int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} xc\varphi(c,x)dxdc + \int_{c=-\infty}^{+\infty}\int_{x=P}^{+\infty} Pc\varphi(c,x)dxdc \\ &= \text{A} + \text{B} - \text{C}\end{aligned} \tag{3}$$

where, A = $\int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} xc\varphi(c,x)dxdc$, B= $\int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty} Pc\varphi(c,x)dxdc$, C = $\int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} Pc\varphi(c,x)dxdc$.

Since $\int_{x=-\infty}^{+\infty} \varphi(c,x)dx$ is the marginal denisty function of c (referred to as h(c)), then

$$\text{B} = P\int_{c=-\infty}^{+\infty} ch(c)dc = P\bar{c} \tag{4}$$

We first expand the simpler term C:

$$\text{C} = \int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} Pc\varphi(c,x)dxdc = P\int_{c=-\infty}^{+\infty} c\{\int_{x=-\infty}^{P} \varphi(c,x)dx\}dc \tag{5}$$

where

$$\int_{x=-\infty}^{P} \varphi(c,x)dx = \int_{x=-\infty}^{P} \frac{1}{2\pi\sigma_c\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[\frac{(c-\bar{c})^2}{\sigma_c^2} - \frac{2\rho(c-\bar{c})(x-\theta)}{\sigma_c\sigma_x} + \frac{(x-\theta)^2}{\sigma_x^2}]} dx \tag{5.1}$$

let $\frac{c-\bar{c}}{\sigma_c} = m$, $\frac{x-\theta}{\sigma_x} = n$, then $dc = \sigma_c dm$, $dx = \sigma_x dn$; when x=P, $n = \frac{P-\theta}{\sigma_x}$. Then eq (5.1) becomes,

$$\int_{x=-\infty}^{P} \varphi(c,x)dx = \int_{n=-\infty}^{\frac{P-\theta}{\sigma_x}} \frac{\sigma_x}{2\pi\sigma_c\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[m^2-2\rho mn+n^2]} dn \tag{5.2}$$

Since $e^{\frac{-1}{2(1-\rho^2)}[m^2-2\rho mn+n^2]} = e^{\frac{-1}{2(1-\rho^2)}[m^2-2\rho mn+n^2+\rho^2m^2-\rho^2m^2]} = e^{\frac{-1}{2(1-\rho^2)}[(n-\rho m)^2+m^2(1-\rho^2)]} = e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}} e^{-\frac{m^2}{2}}$,

eq (5.2) becomes,

$$\int_{x=-\infty}^{P} \varphi(c,x)dx = \frac{e^{-\frac{m^2}{2}}}{2\pi\sigma_c\sqrt{1-\rho^2}} \int_{n=-\infty}^{\frac{P-\theta}{\sigma_x}} e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}} dn \tag{5.3}$$

Let $t = \frac{n-\rho m}{\sqrt{1-\rho^2}}$, then $dn = \sqrt{1-\rho^2}dt$; when $n = \frac{P-\theta}{\sigma_x}$, $t = \frac{\frac{P-\theta}{\sigma_x}-\rho m}{\sqrt{1-\rho^2}}$ =U . Eq (5.3) becomes,

$$\int_{x=-\infty}^{P} \varphi(c,x)dx = \frac{e^{-\frac{m^2}{2}}}{2\pi\sigma_c} \int_{-\infty}^{\mathrm{U}} e^{\frac{-t^2}{2}} dt = \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}\sigma_c} \Phi(\mathrm{U}) \tag{5.4}$$

In the above equation, $\Phi(.)$ is the standard normal cumulative function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt$$

Combining eqs (5.4) and (5), we have

$$\mathrm{C} = P\int_{c=-\infty}^{+\infty} c\frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}\sigma_c}\Phi(\mathrm{U})dc = P\int_{m=-\infty}^{+\infty} (\sigma_c m+\bar{c})\frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}}\Phi(\mathrm{U})dm = \mathrm{C1}+\mathrm{C2} \tag{6}$$

where $\mathrm{C1}=P\bar{c}\int_{m=-\infty}^{+\infty} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}}\Phi(\mathrm{U})dm$ and $\mathrm{C2}=P\sigma_c\int_{m=-\infty}^{+\infty} m\frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}}\Phi(\mathrm{U})dm$.

Since $\Phi(.)$ is unintegrable in eq (6), we have to use a simpler function to approximate $\Phi(.)$.

Now we expand A:

$$\mathrm{A} = \int_{c=-\infty}^{+\infty}\int_{x=-\infty}^{P} \frac{xc}{2\pi\sigma_c\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[\frac{(c-\bar{c})^2}{\sigma_c^2}-\frac{2\rho(c-\bar{c})(x-\theta)}{\sigma_c\sigma_x}+\frac{(x-\theta)^2}{\sigma_x^2}]} dxdc \tag{7}$$

let $\frac{c-\bar{c}}{\sigma_c} = m$, $\frac{x-\theta}{\sigma_x} = n$, then

$$\mathrm{A} = \int_{m=-\infty}^{+\infty}\int_{n=-\infty}^{\frac{P-\theta}{\sigma_x}} (n\sigma_x+\theta)(m\sigma_c+\bar{c})\frac{1}{2\pi\sqrt{1-\rho^2}}e^{\frac{-1}{2(1-\rho^2)}[m^2-2\rho mn+n^2]}dndm$$

$$= \int_{m=-\infty}^{+\infty}\frac{(m\sigma_c+\bar{c})}{2\pi\sqrt{1-\rho^2}}e^{-\frac{m^2}{2}}\left[\int_{n=-\infty}^{\frac{P-\theta}{\sigma_x}}(n\sigma_x+\theta)e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}}dn\right]dm \tag{7.1}$$

let $t=\dfrac{n-\rho m}{\sqrt{1-\rho^2}}$, then

$$\int_{n=-\infty}^{\frac{P-\theta}{\sigma_x}}(n\sigma_x+\theta)e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}}dn = \int_{-\infty}^{\mathrm{U}}\sqrt{1-\rho^2}[(t\sqrt{1-\rho^2}+\rho m)\sigma_x+\theta]e^{\frac{-t^2}{2}}dt$$

$$= (1-\rho^2)\sigma_x\int_{-\infty}^{\mathrm{U}}te^{\frac{-t^2}{2}}dt+\sqrt{1-\rho^2}\sqrt{2\pi}(\sigma_x\rho m+\theta)\int_{-\infty}^{\mathrm{U}}\frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}}dt \tag{7.2}$$

$$= -(1-\rho^2)\sigma_x e^{\frac{-\mathrm{U}^2}{2}}+\sqrt{1-\rho^2}\sqrt{2\pi}(\sigma_x\rho m+\theta)\Phi(\mathrm{U})$$

$\mathrm{A1}=\dfrac{-\sqrt{1-\rho^2}\sigma_x\sigma_c}{2\pi}\displaystyle\int_{m=-\infty}^{+\infty}me^{-\frac{m^2}{2}}e^{-\frac{\mathrm{U}^2}{2}}dm$	$\mathrm{A4}=\dfrac{\sigma_c\theta+\rho\bar{c}\sigma_x}{\sqrt{2\pi}}\displaystyle\int_{m=-\infty}^{+\infty}me^{-\frac{m^2}{2}}\Phi(\mathrm{U})dm$
$\mathrm{A2}=\dfrac{-\sqrt{1-\rho^2}\sigma_x\bar{c}}{2\pi}\displaystyle\int_{m=-\infty}^{+\infty}e^{-\frac{m^2}{2}}e^{-\frac{\mathrm{U}^2}{2}}dm$	$\mathrm{A5}=\dfrac{\bar{c}\theta}{\sqrt{2\pi}}\displaystyle\int_{m=-\infty}^{+\infty}e^{-\frac{m^2}{2}}\Phi(\mathrm{U})dm$
$\mathrm{A3}=\dfrac{\rho\sigma_x\sigma_c}{\sqrt{2\pi}}\displaystyle\int_{m=-\infty}^{+\infty}m^2e^{-\frac{m^2}{2}}\Phi(\mathrm{U})dm$	
$\mathrm{A1_T}=\dfrac{-\sqrt{1-\rho^2}\sigma_x\sigma_c}{2\pi F_{LU}}\displaystyle\int_{\frac{c_L-c}{\sigma_c}}^{\frac{c_U-c}{\sigma_c}}me^{-\frac{m^2}{2}}[e^{-\frac{\mathrm{U}^2}{2}}-e^{-\frac{\mathrm{L}^2}{2}}]dm$	$\mathrm{A4_T}=\dfrac{\sigma_c\theta+\rho\bar{c}\sigma_x}{\sqrt{2\pi}F_{LU}}\displaystyle\int_{\frac{c_L-c}{\sigma_c}}^{\frac{c_U-c}{\sigma_c}}me^{-\frac{m^2}{2}}[\Phi(\mathrm{U})-\Phi(\mathrm{L})]dm$
$\mathrm{A2_T}=\dfrac{-\sqrt{1-\rho^2}\sigma_x\bar{c}}{2\pi F_{LU}}\displaystyle\int_{\frac{c_L-c}{\sigma_c}}^{\frac{c_U-c}{\sigma_c}}e^{-\frac{m^2}{2}}[e^{-\frac{\mathrm{U}^2}{2}}-e^{-\frac{\mathrm{L}^2}{2}}]dm$	$\mathrm{A5_T}=\dfrac{\bar{c}\theta}{\sqrt{2\pi}F_{LU}}\displaystyle\int_{\frac{c_L-c}{\sigma_c}}^{\frac{c_U-c}{\sigma_c}}e^{-\frac{m^2}{2}}[\Phi(\mathrm{U})-\Phi(\mathrm{L})]dm$
$\mathrm{A3_T}=\dfrac{\rho\sigma_x\sigma_c}{\sqrt{2\pi}F_{LU}}\displaystyle\int_{\frac{c_L-c}{\sigma_c}}^{\frac{c_U-c}{\sigma_c}}m^2e^{-\frac{m^2}{2}}[\Phi(\mathrm{U})-\Phi(\mathrm{L})]dm$	

Table 2. Single integrals for A and $\mathrm{A_T}$

Combining eqs (7.1) and (7.2), A becomes

$$\mathrm{A} = \int_{m=-\infty}^{+\infty} \frac{(m\sigma_c + \bar{c})}{2\pi\sqrt{1-\rho^2}} e^{-\frac{m^2}{2}} \left[-(1-\rho^2)\sigma_x e^{-\frac{\mathrm{U}^2}{2}} + \sqrt{1-\rho^2}\sqrt{2\pi}(\sigma_x \rho m + \theta)\Phi(\mathrm{U}) \right] dm \tag{7.3}$$

From eq (7.3), it is straightforward to reduce A into five single integrals, A1 to A5, as follows:

$$\mathrm{A} = \mathrm{A1+A2+A3+A4+A5} \tag{8}$$

Where, A1 to A5 are listed in Table 2.

Combining eqs (3), (4), (6) and (8), we obtain the equation to calculate the revenue when the price and demand are correlated:

$$\text{Revenue} = \mathrm{A1 + A2 + A3 + A4 + A5 + B - C1 - C2} \tag{9}$$

3. Revenue for correlated and truncated price and demand

Besides the independence assumption, in the derivation of the works in the literature[4,6], the integration ranges of price and demand are assumed to be $(-\infty, +\infty)$, which is not the case in the real world. This may bring further discrepancy in revenue computation. To handle this, the formulae for truncated price and demand are derived in this paper and the influence of degree of truncation on revenue computation is studied using case studies. Now suppose that the range of demand is $[x_L, x_U]$, where $-\infty < x_L < x_U < +\infty$ and the range of price is $[c_L, c_U]$, where $-\infty < c_L < c_U < +\infty$. Then the pdf (probability density function) of BBTN (Bivariate Bi-Truncated Normal distribution) is:

$$f_{BBTN}(c,x) = \begin{cases} \dfrac{\varphi(c,x)}{F_{LU}}, & x_L \le x \le x_U \ \ and \ \ c_L \le c \le c_U \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

where, $\varphi(c,x)$ is the pdf of the two-dimensional non-truncated normal distribution function defined by eq (2)

and $F_{LU} = \int_{c_L}^{c_U}\int_{x_L}^{x_U} \varphi(c,x)dxdc$.

We point out that some researchers[14] used incorrect pdf for bivariate left truncated normal distribution as follows

$$f_{BLTN}(c,x) = \begin{cases} \dfrac{\varphi(c,x)}{1-F(x_L,c_L)}, & x_L \le x \text{ and } c_L \le c \\ 0, & \text{otherwise} \end{cases}$$

In fact, the term $1-F(x_L,c_L) = 1-\int_{-\infty}^{c_L}\int_{-\infty}^{x_L} \varphi(c,x)dxdc$ does not equal to $\int_{c_L}^{\infty}\int_{x_L}^{\infty} \varphi(c,x)dxdc$.

Combining eqs (1) and (10), the plant revenue is

$$\text{Revenue} = \text{A}_\text{T} + \text{C}_\text{T} \tag{11}$$

where, $\text{A}_\text{T} = \int_{c=c_L}^{c_U}\int_{x=x_L}^{P} xcf_{BBTN}(c,x)dxdc$; $\text{C}_\text{T} = \int_{c=c_L}^{c_U}\int_{x=P}^{x_U} Pcf_{BBTN}(c,x)dxdc$ **.**

Here, the value of P should locate in [x_L, x_U].

Now we expand A_T:

$$\text{A}_\text{T} = \frac{1}{F_{LU}}\int_{c=c_L}^{c_U}\int_{x=x_L}^{P} \frac{xc}{2\pi\sigma_c\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[\frac{(c-\bar{c})^2}{\sigma_c^2}-\frac{2\rho(c-\bar{c})(x-\theta)}{\sigma_c\sigma_x}+\frac{(x-\theta)^2}{\sigma_x^2}]} dxdc \tag{12}$$

substitute c and x with m and n respectively, eq (12) becomes

$$\text{A}_\text{T} = \int_{m=\frac{c_L-\bar{c}}{\sigma_c}}^{\frac{c_U-\bar{c}}{\sigma_c}} \frac{(m\sigma_c+\bar{c})}{2\pi F_{LU}\sqrt{1-\rho^2}} e^{-\frac{m^2}{2}} \left[\int_{n=\frac{x_L-\theta}{\sigma_x}}^{\frac{P-\theta}{\sigma_x}} (n\sigma_x+\theta)e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}} dn\right] dm \tag{12.1}$$

substitute n with t and define $\text{L} = \dfrac{\frac{x_L-\theta}{\sigma_x}-\rho m}{\sqrt{1-\rho^2}}$:

$$\int_{n=\frac{x_L-\theta}{\sigma_x}}^{\frac{P-\theta}{\sigma_x}} (n\sigma_x+\theta)e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}}dn = \int_{\mathrm{L}}^{\mathrm{U}} \sqrt{1-\rho^2}[(t\sqrt{1-\rho^2}+\rho m)\sigma_x+\theta]e^{\frac{-t^2}{2}}dt$$

$$= -(1-\rho^2)\sigma_x(e^{\frac{-\mathrm{U}^2}{2}}-e^{\frac{-\mathrm{L}^2}{2}})+\sqrt{1-\rho^2}\sqrt{2\pi}(\sigma_x\rho m+\theta)(\Phi(\mathrm{U})-\Phi(\mathrm{L})) \quad (12.2)$$

Combing eqs (12.1) and (12.2), $\mathrm{A_T}$ becomes

$$\mathrm{A_T} = \int_{\frac{c_L-\bar{c}}{\sigma_c}}^{\frac{c_U-\bar{c}}{\sigma_c}} \frac{(m\sigma_c+\bar{c})}{2\pi F_{LU}\sqrt{1-\rho^2}}e^{-\frac{m^2}{2}}\left[-(1-\rho^2)\sigma_x(e^{-\frac{\mathrm{U}^2}{2}}-e^{\frac{-\mathrm{L}^2}{2}})+\sqrt{1-\rho^2}\sqrt{2\pi}(\sigma_x\rho m+\theta)(\Phi(\mathrm{U})-\Phi(\mathrm{L}))\right]dm \quad (12.3)$$

From eq (12.3), it is straightforward to reduce $\mathrm{A_T}$ into five single integrals, $\mathrm{A1_T}$ to $\mathrm{A5_T}$, as follows:

$$\mathrm{A_T} = \mathrm{A1_T} + \mathrm{A2_T} + \mathrm{A3_T} + \mathrm{A4_T} + \mathrm{A5_T} \quad (13)$$

Where $\mathrm{A1_T}$ to $\mathrm{A5_T}$ are listed in Table 2.

Now we expand $\mathrm{C_T}$:

$$\mathrm{C_T} = \int_{c=c_L}^{c_U}\int_{x=P}^{x_U} Pcf_{BBTN}(c,x)dxdc = \frac{P}{F_{LU}}\int_{c=c_L}^{c_U} c\{\int_{x=P}^{x_U}\varphi(c,x)dx\}dc \quad (14)$$

substitute *c* and *x* with *m* and *n* respectively

$$\int_{x=P}^{x_U}\varphi(c,x)dx = \frac{e^{-\frac{m^2}{2}}}{2\pi\sigma_c\sqrt{1-\rho^2}}\int_{n=\frac{P-\theta}{\sigma_x}}^{\frac{x_U-\theta}{\sigma_x}} e^{\frac{-(n-\rho m)^2}{2(1-\rho^2)}}dn \quad (14.1)$$

substitute *n* with *t* and define $\mathrm{U_U} = \dfrac{\frac{x_U-\theta}{\sigma_x}-\rho m}{\sqrt{1-\rho^2}}$:

$$\int_{x=P}^{x_U}\varphi(c,x)dx = \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}\sigma_c}(\Phi(\mathrm{U_U})-\Phi(\mathrm{U})) \quad (14.2)$$

Combining eqs (14) and (14.2), we have

$$\mathrm{C_T} = \mathrm{C1_T} + \mathrm{C2_T} \quad (15)$$

where $\mathrm{C1_T} = \dfrac{P\bar{c}}{\sqrt{2\pi}F_{LU}}\displaystyle\int_{m=\frac{c_L-\bar{c}}{\sigma_c}}^{\frac{c_U-\bar{c}}{\sigma_c}} e^{-\frac{m^2}{2}}[\Phi(\mathrm{U_U})-\Phi(\mathrm{U})]dm$, $\mathrm{C2_T} = \dfrac{P\sigma_c}{\sqrt{2\pi}F_{LU}}\displaystyle\int_{m=\frac{c_L-\bar{c}}{\sigma_c}}^{\frac{c_U-\bar{c}}{\sigma_c}} me^{-\frac{m^2}{2}}[\Phi(\mathrm{U_U})-\Phi(\mathrm{U})]dm$.

Combining eqs (11), (13) and (15), we obtain the equation to calculate the revenue when the price and demand are correlated and truncated:

$$\text{Revenue} = A1_T + A2_T + A3_T + A4_T + A5_T + C1_T + C2_T \tag{16}$$

4. Revenue calculation using approximated method

4.1. Approximation of the standard normal cumulative function

We still cannot calculate the single integrals listed in Tables 2 and 3 directly because the standard normal cumulative function, Φ(.), is unintegrable. To overcome this difficulty, we should use some simpler functions to approximate Φ(.). There exist some accurate approximations to the standard normal cumulative function in the literature[34]. However, those approximations, complicated by exponential functions, are still unintegrable or too complicated to integrate. In this paper, simple polynomial functions are used to approximate Φ(.):

$$\Phi(x) = a_0 + a_1 x + a_3 x^3 + a_5 x^5 + ... + a_n x^n, \qquad n=1,2,3.... \tag{17}$$

Table 3 lists the regressed coefficients a_0~a_n when seventh-order and ninth-order polynomial functions are used (n =7 and 9 in eq (17)) and the regression ranges of x are [–3, 3] and [–5, 5], respectively. Note that, the regression range of x is selected based on the range of U, L and U^U (see Appendix II for details). The last column is the sum of error square of the regression. The accuracy of the approximation will be shown in case studies.

order	Range of x	a_0	a_1	a_3	a_5	a_7	a_9	Sum of error square
7th	[-3, 3]	0.5	0.3942473009	-0.0581252700	0.0056884266	-2.28133E-04	NA	6.99847E-05
	[-5, 5]	0.5	0.3587807973	-0.0348111040	0.0017667944	-3.18160E-05	NA	0.0134016761
9th	[-3, 3]	0.5	0.3979964902	-0.064081932	0.0082040258	-6.17138E-04	1.9877E-05	2.05180E-06
	[-5, 5]	0.5	0.3814916121	-0.047908688	0.0037750801	-1.44598E-04	2.0933E-06	0.0019402717

Table 3 Coefficients of polynomial approximation functions for Φ(.)

4.2. Revenue calculation using approximated Φ(.)

With Φ(.) approximated by eq (17), we can further derive the formulae for the single integrals. Here we show formulae for seventh-order polynomial approximation. It is straightforward to extend to ninth-order polynomial approximation.

4.2.1 Revenue for correlated price and demand

We first define:

$$W = \frac{-\rho}{\sqrt{1-\rho^2}}, \quad Z_U = \frac{P-\theta}{\sigma_x\sqrt{1-\rho^2}}$$

Applying eq (17):

$$\begin{aligned}\Phi(\mathrm{U}) &= \Phi\left(\frac{\frac{P-\theta}{\sigma_x} - \rho m}{\sqrt{1-\rho^2}}\right) = \Phi(Wm + Z_U) \\ &= a_0 + a_1(Wm+Z_U) + a_3(Wm+Z_U)^3 + a_5(Wm+Z_U)^5 + a_7(Wm+Z_U)^7\end{aligned} \tag{18}$$

expand eq (18):

$$\Phi(\mathrm{U}) = \mathrm{F+G+H+I+J+K+L+M} \tag{19}$$

where, $\mathrm{F} = a_7 W^7 m^7$, $\mathrm{G} = 7a_7 W^6 Z_U m^6$, $\mathrm{H} = (a_5 W^5 + 21a_7 W^5 Z_U^2)m^5$, $\mathrm{I} = (5a_5 W^4 Z_U + 35a_7 W^4 Z_U^3)m^4$,

$\mathrm{J} = (a_3 W^3 + 10a_5 W^3 Z_U^2 + 35a_7 W^3 Z_U^4)m^3$, $\mathrm{K} = (3a_3 W^2 Z_U + 10a_5 W^2 Z_U^3 + 21a_7 W^2 Z_U^5)m^2$,

$\mathrm{L} = (a_1 W + 3a_3 W Z_U^2 + 5a_5 W Z_U^4 + 7a_7 W Z_U^6)m$, $\mathrm{M} = a_0 + a_1 Z_U + a_3 Z_U^3 + a_5 Z_U^5 + a_7 Z_U^7$.

With eq (19), the single integrals for A are now integrable and listed in Table 4. In Table 4, coefficients F to M are defined in eq (19); E_1, E_3, M_0 to M_8 are derived in Appendix III.

$\mathrm{A1} = \dfrac{-\sqrt{1-\rho^2}\,\sigma_x\sigma_c}{2\pi}\mathrm{E}_1$
$\mathrm{A2} = \dfrac{-\sqrt{1-\rho^2}\,\sigma_x\bar{c}}{2\pi}\mathrm{E}_3$
$\mathrm{A3} = \dfrac{\rho\sigma_x\sigma_c}{\sqrt{2\pi}}(\mathrm{G}M_8 + \mathrm{I}M_6 + \mathrm{K}M_4 + \mathrm{M}M_2)$
$\mathrm{A4} = \dfrac{\sigma_c\theta + \rho\bar{c}\sigma_x}{\sqrt{2\pi}}(\mathrm{F}M_8 + \mathrm{H}M_6 + \mathrm{J}M_4 + \mathrm{L}M_2)$
$\mathrm{A5} = \dfrac{\bar{c}\theta}{\sqrt{2\pi}}(\mathrm{G}M_6 + \mathrm{I}M_4 + \mathrm{K}M_2 + \mathrm{M}M_0)$
$\mathrm{C1} = \dfrac{P\bar{c}}{\sqrt{2\pi}}(\mathrm{G}M_6 + \mathrm{I}M_4 + \mathrm{K}M_2 + \mathrm{M}M_0)$
$\mathrm{C2} = \dfrac{P\sigma_c}{\sqrt{2\pi}}(\mathrm{F}M_8 + \mathrm{H}M_6 + \mathrm{J}M_4 + \mathrm{L}M_2)$
$\mathrm{A1}_\mathrm{T} = \dfrac{-\sqrt{1-\rho^2}\,\sigma_x\sigma_c}{2\pi F_{LU}}(\mathrm{E}_1' - \mathrm{E}_2')$
$\mathrm{A2}_\mathrm{T} = \dfrac{-\sqrt{1-\rho^2}\,\sigma_x\bar{c}}{2\pi F_{LU}}(\mathrm{E}_3' - \mathrm{E}_4')$
$\mathrm{A3}_\mathrm{T} = \dfrac{\rho\sigma_x\sigma_c}{\sqrt{2\pi}F_{LU}}(\mathrm{G_T}M_8' + \mathrm{H_T}M_7' + \mathrm{I_T}M_6' + \mathrm{J_T}M_5' + \mathrm{K_T}M_4' + \mathrm{L_T}M_3' + \mathrm{M_T}M_2')$
$\mathrm{A4}_\mathrm{T} = \dfrac{\sigma_c\theta + \rho\bar{c}\sigma_x}{\sqrt{2\pi}F_{LU}}(\mathrm{G_T}M_7' + \mathrm{H_T}M_6' + \mathrm{I_T}M_5' + \mathrm{J_T}M_4' + \mathrm{K_T}M_3' + \mathrm{L_T}M_2' + \mathrm{M_T}M_1')$
$\mathrm{A5}_\mathrm{T} = \dfrac{\bar{c}\theta}{\sqrt{2\pi}F_{LU}}(\mathrm{G_T}M_6' + \mathrm{H_T}M_5' + \mathrm{I_T}M_4' + \mathrm{J_T}M_3' + \mathrm{K_T}M_2' + \mathrm{L_T}M_1' + \mathrm{M_T}M_0')$
$\mathrm{C1}_\mathrm{T} = \dfrac{P\bar{c}}{\sqrt{2\pi}F_{LU}}(\mathrm{G_T'}M_6' + \mathrm{H_T'}M_5' + \mathrm{I_T'}M_4' + \mathrm{J_T'}M_3' + \mathrm{K_T'}M_2' + \mathrm{L_T'}M_1' + \mathrm{M_T'}M_0')$
$\mathrm{C2}_\mathrm{T} = \dfrac{P\sigma_c}{\sqrt{2\pi}F_{LU}}(\mathrm{G_T'}M_7' + \mathrm{H_T'}M_6' + \mathrm{I_T'}M_5' + \mathrm{J_T'}M_4' + \mathrm{K_T'}M_3' + \mathrm{L_T'}M_2' + \mathrm{M_T'}M_1')$

Table 4. The single integrals for eqs (9) and (16)

4.2.2 Revenue for correlated and truncated price and demand

We first define:

$$Z_L = \frac{x_L - \theta}{\sigma_x\sqrt{1-\rho^2}}, \quad Z_{UU} = \frac{x_U - \theta}{\sigma_x\sqrt{1-\rho^2}}$$

then

$$\Phi(L)=\Phi(\frac{\frac{x_L-\theta}{\sigma_x}-\rho m}{\sqrt{1-\rho^2}})=\Phi(Wm+Z_L)\,;\quad \Phi(\mathrm{U_U})=\Phi(\frac{\frac{x_U-\theta}{\sigma_x}-\rho m}{\sqrt{1-\rho^2}})=\Phi(Wm+Z_{UU})$$

Expanding $\Phi(\mathrm{U})$ and $\Phi(\mathrm{L})$, we obtain:

$$\Phi(\mathrm{U})-\Phi(\mathrm{L})=\Phi(Wm+Z_U)-\Phi(Wm+Z_L)=\mathrm{G_T}+\mathrm{H_T}+\mathrm{I_T}+\mathrm{J_T}+\mathrm{K_T}+\mathrm{L_T}+\mathrm{M_T} \qquad (20)$$

where $\mathrm{G_T}=7a_7W^6(Z_U-Z_L)m^6$, $\mathrm{H_T}=21a_7W^5(Z_U^2-Z_L^2)m^5$, $\mathrm{I_T}=[5a_5W^4(Z_U-Z_L)+35a_7W^4(Z_U^3-Z_L^3)]m^4$,

$\mathrm{J_T}=[10a_5W^3(Z_U^2-Z_L^2)+35a_7W^3(Z_U^4-Z_L^4)]m^3$,

$\mathrm{K_T}=[3a_3W^2(Z_U-Z_L)+10a_5W^2(Z_U^3-Z_L^3)+21a_7W^2(Z_U^5-Z_L^5)]m^2$,

$\mathrm{L_T}=[3a_3W(Z_U^2-Z_L^2)+5a_5W(Z_U^4-Z_L^4)+7a_7W(Z_U^6-Z_L^6)]m$,

$\mathrm{M_T}=a_1(Z_U-Z_L)+a_3(Z_U^3-Z_L^3)+a_5(Z_U^5-Z_L^5)+a_7(Z_U^7-Z_L^7)$.

Expanding $\Phi(\mathrm{U})$ and $\Phi(\mathrm{U_U})$, we obtain:

$$\Phi(\mathrm{U_U})-\Phi(\mathrm{U})=\Phi(Wm+Z_{UU})-\Phi(Wm+Z_U)=\mathrm{G_T'}+\mathrm{H_T'}+\mathrm{I_T'}+\mathrm{J_T'}+\mathrm{K_T'}+\mathrm{L_T'}+\mathrm{M_T'} \qquad (21)$$

where

$\mathrm{G_T'}=7a_7W^6(Z_{UU}-Z_U)m^6$, $\mathrm{H_T'}=21a_7W^5(Z_{UU}^2-Z_U^2)m^5$, $\mathrm{I_T'}=[5a_5W^4(Z_{UU}-Z_U)+35a_7W^4(Z_{UU}^3-Z_U^3)]m^4$,

$\mathrm{J_T'}=[10a_5W^3(Z_{UU}^2-Z_U^2)+35a_7W^3(Z_{UU}^4-Z_U^4)]m^3$,

$\mathrm{K_T'}=[3a_3W^2(Z_{UU}-Z_U)+10a_5W^2(Z_{UU}^3-Z_U^3)+21a_7W^2(Z_{UU}^5-Z_U^5)]m^2$,

$\mathrm{L_T'}=[3a_3W(Z_{UU}^2-Z_U^2)+5a_5W(Z_{UU}^4-Z_U^4)+7a_7W(Z_{UU}^6-Z_U^6)]m$,

$\mathrm{M_T'}=a_1(Z_{UU}-Z_U)+a_3(Z_{UU}^3-Z_U^3)+a_5(Z_{UU}^5-Z_U^5)+a_7(Z_{UU}^7-Z_U^7)$.

With eqs (20) and (21), the single integrals for $\mathrm{A_T}$ are now integrable and listed in Table 4. In Table 4, $\mathrm{E_1'}$ to $\mathrm{E_4'}$,

M_0' to M_8' are derived in Appendix IV. Some properties of the revenue are derived In Appendix V.

5. Impact of customer service levels

Customer service level is an important index that must be monitored and maintained[5]. Two types of service levels (or customer satisfaction levels) are commonly used in the industry. The Type 1 service level (usually also called the confidence level) is the probability of not stocking out in all scenarios or horizons[35]. The Type 2 service level (also often called the fill rate) is the proportion of demands that are met from a plant[35]. The Type 1 service level is widely applied in chance constrained programming up to now. However, it is not how service is interpreted in most applications[35]. The Type 2 service level is a greater concern of most managers in industry[6,35]. The difference of Type 1 and 2 service level can be found in the literature[6,35].

5.1 Type 1 service level

To apply Type 1 service level on uncertain customer demand *x*, the following constraint should be added in the model[2,6]:

$$\Pr\left(D \geq x\right) \geq \alpha$$

where α is the Type 1 service level or confidence level target, *D* is the production rate (no inventory) or the deliverable amount (with inventory). The above constraint is transformed to the following by applying chance constrained programming[2,6]:

$$D \geq F^{-1}(\alpha) \tag{22}$$

In the above constraint, F^{-1} is the reverse cumulative distribution function of the product demand. When demand conforms to non-truncated normal distribution, constraint (22) becomes:

$$D \geq \theta + \sigma_x \Phi^{-1}(\alpha) \tag{22a}$$

When demand conforms to doubly truncated normal distribution, constraint (22) becomes:

$$\Pr\left(D \geq x\right) = \int_{x_L}^{D} \rho_{BTN}(x)dx \geq \alpha \tag{22b}$$

where $\rho_{BTN}(x)$ is the bi-truncated density function of demand *x*:

$$\rho_{BTN}(x) = \frac{\phi(\frac{x-\theta}{\sigma_x})}{\sigma_x[\Phi(\frac{x_U-\theta}{\sigma_x}) - \Phi(\frac{x_L-\theta}{\sigma_x})]} = \frac{\phi(\frac{x-\theta}{\sigma_x})}{\sigma_{BTN}}.$$

where $\sigma_{BTN} = \sigma_x[\Phi(Z_{XU}) - \Phi(Z_{XL})]$ and $Z_{XU} = \frac{x_U-\theta}{\sigma_x}$, $Z_{XL} = \frac{x_L-\theta}{\sigma_x}$.

let $t = \frac{x-\theta}{\sigma_x}$ and define $Z_{XP} = \frac{D-\theta}{\sigma_x}$, then

$$\int_{x_L}^{D} \rho_{BTN}(x)dx = \frac{\sigma_x}{\sigma_{BTN}} \int_{Z_{XL}}^{Z_{XP}} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = \frac{\sigma_x}{\sigma_{BTN}}[\Phi(Z_{XP}) - \Phi(Z_{XL})] = \frac{\Phi(Z_{XP}) - \Phi(Z_{XL})}{\Phi(Z_{XU}) - \Phi(Z_{XL})}$$

Thus, (22b) becomes

$\frac{\Phi(Z_{XP}) - \Phi(Z_{XL})}{\Phi(Z_{XU}) - \Phi(Z_{XL})} \geq \alpha$, or $\Phi(Z_{XP}) \geq \Phi(Z_{XL}) + \alpha(\Phi(Z_{XU}) - \Phi(Z_{XL}))$, i.e.,

$$D \geq \theta + \sigma_x \Phi^{-1}(\Phi(Z_{XL}) + \alpha(\Phi(Z_{XU}) - \Phi(Z_{XL}))) \quad (22c)$$

5.2 Type 2 service level

Type 2 service level is the one that most managers need[6,35]. However, it cannot be accurately approximated by a Type 1 service level[6]. To apply Type 2 service level on customer demand, the following constraint should be added in the model[6]:

$$\frac{S}{\theta} \geq \beta \quad (23)$$

where β is the Type 2 service level or fill rate target defined by the decision maker, S is the actual amount of product sold to customers. S/θ represents the actual Type 2 service level. For non-truncated normally distributed demand, the standard loss function, *LF*(.), has been effectively applied and approximated to compute S in the literature[6]. In this paper, we extend the research to cases when demand is truncated.

5.3. The actual Type 2 service level

When demand x is doubly truncated, the actual amount of product sold to customers, S_{BTN}, is the minimum value of the customer demand and the production rate:

$$S_{BTN} = \int_{x_L}^{x_U} \min\{P,\ x\}\rho_{BTN}(x)dx \qquad (24)$$

Expand eq (24), we have

$$S_{BTN} = \int_{x_L}^{x_U} x\rho_{BTN}(x)dx - \int_{P}^{x_U} (x-P)\rho_{BTN}(x)dx = \theta_{BTN} - LF_{BTN}(P) \qquad (25)$$

where θ_{BTN} and $LF_{BTN}(P)$ are the mean and the loss function of the bi-truncated normal demand, respectively. Loss function represents the amount of unmet demand (the backorder level) of a plant facing uncertain demand[36]. θ_{BTN} and $LF_{BTN}(P)$ are derived in Appendix VI. From eqs (23) and (25), the actual Type 2 service level for bi-truncated normal demand is:

$$\frac{S_{BTN}}{\theta_{BTN}} = \frac{\theta_{BTN} - LF_{BTN}(P)}{\theta_{BTN}} = 1 - \frac{LF_{BTN}(P)}{\theta_{BTN}} \qquad (26)$$

Eq (23) then becomes:

$$\frac{S_{BTN}}{\theta_{BTN}} = 1 - \frac{LF_{BTN}(P)}{\theta_{BTN}} \ge \beta \qquad (27)$$

6. The optimization model

The optimization model implementing the revenue calculation and the Type 1 and 2 service levels is shown as follows:

$$\begin{aligned}
&\max \ \sum_{\mathrm{p}} \mathrm{Revenue}_{\mathrm{p}} - \mathrm{COSTS}\ (\text{eqs 9 or 16}) \\
&\text{s.t.}\quad \text{Constraints for } \mathrm{Revenue}_{\mathrm{p}}\ (\text{Tables 5 and 6}) \\
&\qquad\ \ \text{Constraints for Type 1 or 2 service levels (eqs 22 and 27)} \\
&\qquad\ \ \text{Constraints for COSTS} \\
&\qquad\ \ \text{Other operational constraints}
\end{aligned}$$

where $\mathrm{Revenue}_{\mathrm{p}}$ is revenue from product p. The model is to maximize the net profit of a plant. We use the formulae (eqs (9) or (16)) derived in this paper to calculate the actual revenue for each product. Tables 5 and 6 are used to support eqs (9) or (16). The constraints for Type 1 or 2 service levels are eqs (22) and (27), respectively. COSTS includes the raw material cost, operating cost, inventory cost and investment cost, etc.

Other operational constraints, such as material balance and product quality, should also be included. Parameters such as W, Z_L, Z_{UU}, Z_{XU}, Z_{XL}, M_0 to M_8, $M_0^{'}$ to $M_8^{'}$, F_{LU}, θ_{BTN} etc. should be calculated before running the optimization model according to their definitions and the formulae derived in Appendixes III~VI.

7. Case Study

Two examples are used to illustrate the influences of correlation and truncation of price and demand on plant revenue. Example 1 is taken from the case 1 of Li et al.[6]. Figure 3[6] shows the configuration of example 1. MTBE and GASO (gasoline) enter the gasoline blending (GB) unit to produce two products: 90# (GASO'90) and 93# (GASO'93) gasoline. The price of GASO and MTBE are 1400 and 3500 Yuan/tom, respectively. The price of 90# and 93# gasoline are 3215 and 3387 Yuan/ton, respectively. The means of the demand for 90# and 93# gasoline are 50 and 70 tons, respectively**.** The octane number of GASO and MTBE are 70 and 101, respectively. The octane number of 90# and 93# gasoline are 90 and 93, respectively. The blending requirement is that the octane number of each product should equal or be greater than the required octane number of that product. No inventory is considered and the overproduced products are assumed to be valueless. All examples are formulated with GAMS[37]. The solver MINOS5 in GAMS 21.7 is used for NLP.

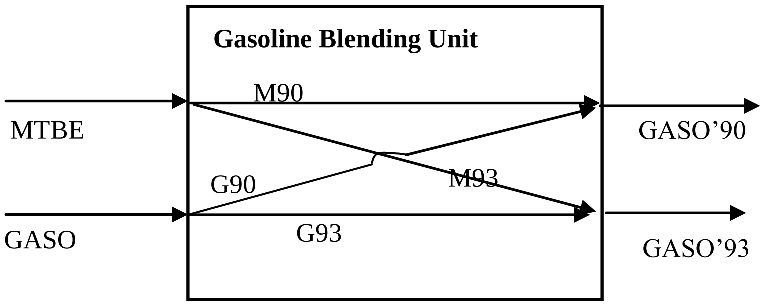

Figure 3 The configuration of example 1

10.1 Accuracy of different approximation functions

In this case, the correlation coefficients between all products and their prices are set to zero and hence the normally distributed price and demand becomes independent. The results from eq (9) should then be the same as that obtained from the literature[6]. The standard deviation of 90# and 93# gasoline demands are both 10 tons. In Table 5.1, the result of the first row is taken from the case 1 of Li et al.[6] Rows 2 to 4 are results from Φ(.) approximation methods with three orders (5^{th}, 7^{th}, 9^{th}). It can be seen that, all three approximation methods have rather high accuracy. The polynomial approximation with higher order has higher accuracy. For the sake of simplicity, seventh order polynomial function is used to approximate the standard normal cumulative function in this paper because it has high enough accuracy. Our formulae involve more equations and variables and thus, the solution time is longer than that in the literature[6]. In this small case study, the solution times of all methods are similar.

	Net Profit	Gap (%)	Solution time (s)	SINGLE EQUATIONS	SINGLE VARIABLES
Li et al. 2004	38733.69		0.03	11	13
5^{th} polynomial	38882.36	0.38	0.04	37	39
7^{th} polynomial	38685.95	-0.12	0.05	41	43
9^{th} polynomial	38717.32	-0.04	0.05	47	49

Table 5.1 Accuracies of different polynomial functions

In Table 5.2, the revenue calculated from eq (9) using GAMS for 90# gasoline is compared with the result by integrating eq (3) directly using MATLAB®[38] at different correlation coefficients. The price and demand are non-truncated and the standard deviation of 90# gasoline is 300 Yuan/ton**.** The production rate of 90# gasoline is fixed to 39.565 at all correlation coefficients. MATLAB® uses rigorous double integral to compute revenue and thus accurate. We can see that the gap between the approximation formulae and MATLAB® is very small. It is difficult to implement MATLAB® into refinery planning model because MATLAB® takes very long time to compute eq (3) and it is difficult to iteratively communicate MATLAB® with an optimization solver.

ρ	MATLAB	7^{th} polynomial approximation	Gap,%
0.0	1.24740E+05	1.24720E+05	-0.016
0.1	1.24780E+05	1.24760E+05	-0.016
0.2	1.24830E+05	1.24810E+05	-0.016
0.3	1.24870E+05	1.24850E+05	-0.016
0.4	1.24920E+05	1.24900E+05	-0.016
0.5	1.24960E+05	1.24950E+05	-0.008

Table 5.2 Comparison with MATLAB

10.2. Effect of correlation

In this case, we consider the influence of correlation coefficient on plant revenue at different CVs (Coefficient of Variation, the ratio of the standard deviation σ to the mean μ : CV=σ / μ). The standard deviation of 90# and 93# gasoline at different CVs are listed in Table 6. The standard deviation of 90# and 93# gasoline prices at different CVs are assumed to be fixed at 600 and 620 Yuan/ton, respectively.

	CV		
Products	0.2	0.3	0.5
90# gasoline	10	15	25
93# gasoline	10	20	35

Table 6 Standard deviation of products at different CVs for example 1

ρ	Net Profit, Yuan	Difference,%
0	10097.50	0.0
0.1	10589.83	4.9
0.2	11096.87	9.9
0.3	11638.74	15.3
0.4	12225.64	21.1
0.5	12853.69	27.3

Table 7 Effect of correlation at CV=0.5 in example 1

The net profits at different correlation coefficients at CV=0.5 are listed in Table 7 (the correlation coefficients between all products and their demands are set to be the same). It can be seen that, the net profit at correlation coefficient of 0.4 (near the real world data) is 21.1% higher than the net profit calculated by assuming independent demand and price (correlation coefficient=0.0). That means, if for a large enough CV of a product, assuming independent price and demand may underestimate the net profit by up to 21%. Note that when the correlation coefficient is too high (bigger than 0.6), the net profit will decrease significantly.

In general, the revenue difference between the independent and correlated cases depends on the CV of products. In the problem studied here, if the CV of a product takes value of 0.2, the net profit difference between $\rho=0.4$ and $\rho=0$ is about 2%. This difference is about 5% for CV of 0.3. We set $\rho=0.4$ for correlated demand and price because it near the real world value according to the regressed data from *EIA*[9]. If ρ is set to be a constant, then as the standard deviation of price increases, the revenue increases slightly. However, as the standard deviation of demand increases, the revenue decreases significantly.

10.3 Effect of Truncation

Integrating over the whole range of a normal distribution may give incorrect results to revenue calculation. The formulae derived for bivariate double-truncated normal distribution are applied in the model. The results are shown in Tables 10 and 11 (the product production rates of truncated cases are fixed to those of the non-truncated case for fair comparison). In Table 8 (at CV=0.2), it can be seen that, if the price and demand vary inside two standard deviations of their mean values, integrating over the whole range will underestimate the revenue by 2 to 3%. If the price and demand vary inside one standard deviation of their mean values, integrating over the whole range will underestimate the revenue by about 12%. The revenue difference between truncated and non-truncated becomes much more significant for large enough CV. In Table 8 (at CV=0.5), integrating over the whole range will underestimate the revenue from 20% up to 130%.

	CV=0.2			CV=0.5		
	Integration Range of Non-truncated case	Integration Ranges of Truncated Cases		Integration Range of Non-truncated case	Integration Ranges of Truncated Cases	
ρ	(–∞, +∞)	(μ+/–2σ)	(μ+/–σ)	(–∞, +∞)	(μ+/–2σ)	(μ+/–σ)
0	38685.9	39847.6	43513.6	10097.5	14986.6	23935.1
0.1	38851.2	39948.7	43772.7	10589.8	15234.4	24545.5
0.2	39021.0	40107.0	43958.2	11096.9	15556.8	25030.0
0.3	39202.3	40418.5	44166.1	11638.7	16067.2	25457.7
0.4	39398.5	40792.3	44369.3	12225.6	16693.6	25907.6
0.5	39608.6	39418.8	44445.2	12853.7	15504.5	26286.5

Table 8 Effect of truncation at CV=0.2 and CV=0.5 in example 1

10.4 Effect of customer service levels

Table 9 lists the net profits when the constraints for Type 1 and 2 service levels are added into the model. The price and demand vary inside two standard deviations of their mean values. The CVs of demand and price are set to 0.2. The net profits in the second column (No C.L./F.L.) are slightly higher than those in the third column of Table 8 because the product production rates are free to change in this case.

It can be seen that, the net profit decreases when we set Type 1 service level target to 0.5. This is because the production rate has to be greater than certain value to satisfy the customer demand. Even though a 50% of confidence level seems to be low, the actual fill rates achieved are high enough: 92.% and 94.8% for product 1 and 2, respectively. In this case, the gap between these two service levels is rather big. When fill rate is set to 0.9, the net profit decreases by about 7~10% compare to “No C.L./F.L.” case. In this case, the net profit becomes negative (-38408.2 Yuan at ρ=0.4) when we set too high C.L. target (e.g., 95%). The plant loses money if the customer demand has to be satisfied at too high ratio. A planning strategy taking Type 1 service level target into account only might be thus suboptimal. A plant has to compromise between the net profit and the two service levels. In fact, when C.L. is 95%, the actual fill rates for product 1 and 2 are both 99.9%, which are unnecessarily high. In this case, setting a Type 2 service level target, F.L.=0.9, is a better choice.

ρ	No C.L./F.L.	C.L.=0.5	F.L.=0.90
0	39885.4	28469.6	35868.0
0.1	39980.6	28877.1	36136.1
0.2	40135.6	29373.0	36478.4
0.3	40445.1	30052.0	36987.2
0.4	40817.8	30805.2	37557.8
0.5	39439.4	29713.7	36318.7

No C.L./F.L.: No service level constraints are added;
C.L.: Type 1 service level constraint is added;
F.L.: Type 2 service level constraint is added.
Table 9 Net profit at different service levels at CV=0.2 in example 1

10.5 Example 2

A larger size example (example 2) was used to illustrate the results using our derived formula. A process flow diagram of a refinery plant is shown in Figure 4. Details of this example can be found in Li et al.[39,40]. The problem contains three main production units: CDU (Crude Distillation Unit), GB (Gasoline Blending) and DB (Diesel Oil Blending). Crude oil is separated into three fractions by CDU. Gasoline and MTBE enter the GB to produce two products: 90# gasoline and 93# gasoline. Diesel oil and naphtha enter the DB to produce another two products: -10# diesel and 0# diesel. The prices (Yuan/ton) of the raw materials and the products are listed in Table 10. The capacity of the CDU is 400 ton/day; the CDU operation cost is 20 Yuan/ton/day. The CDU transfer ratios of crude oil to gasoline, to diesel oil, and to naphtha are fixed at 0.2, 0.3 and 0.5, respectively. The market demands for 90# gasoline and 93# gasoline are assumed to conform to normal distributions, N(50, 25) and N(40, 25), respectively. For the sake of simplicity, the market demands for the other two products are assumed to be deterministic without losing the generality of the model. The objective of this case study is to determine the optimal planning strategy for the refinery under uncertain market conditions. The results are shown in Table 11.

In Table 11, it can be seen that, as ρ increases, the net profit increases along each column. In the second and third column, confidence level and fill rate are considered and assumed to be 95%. The net profit reduced by around 30% and 12% when C.L. and F.L. are set to 95%, respectively. In the truncated case, the net profit increases by 1.4% to 4.5% when the degree of truncation is (μ+/–2σ). The net profit increases by 14.8% to 22.3% when the degree of truncation is (μ+/–σ).

Raw Material		Products			
Crude Oil	MTBE	90# gasoline	93# gasoline	-5# diesel	0# diesel
1400	3500	3215	3387	2700	2500

Table 10 Price Data (Yuan/ton) for example 2

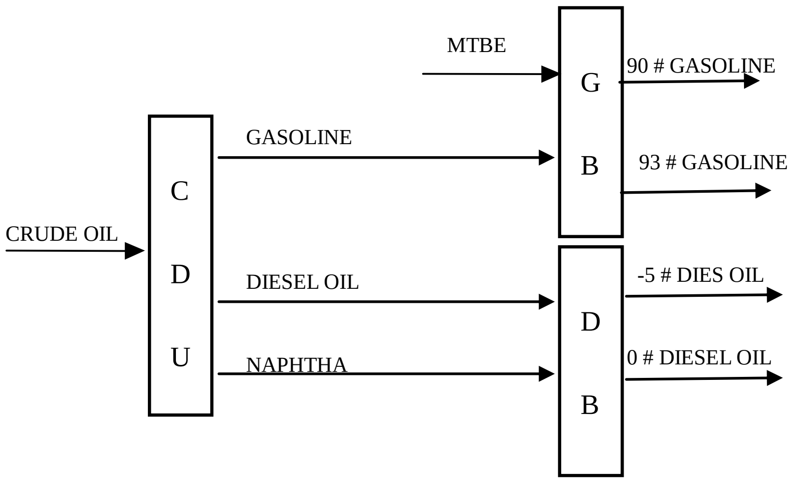

Figure 4 The configuration of example 2

	Non-truncated			Truncated	
ρ	No. C.L./F.L.	C.L.=0.95	F.L.=0.95	(μ+/–2σ)	(μ+/–σ)
0	120,118.8	81,967.0	102,912.8	125,549.6	146,859.5
0.1	121,971.4	84,814.5	105,452.8	126,730.4	147,108.9
0.2	123,843.1	87,698.7	107,975.3	128,044.3	147,388.0
0.3	125,730.8	90,600.8	110,494.3	129,586.9	147,776.8
0.4	127,632.7	93,495.6	113,026.5	131,271.3	148,242.2
0.5	129,551.4	96,490.6	115,577.7	131,312.3	148,690.1

Table 11 Results from example 2

12. Conclusion

In this paper, the correlation between price and demand as well as their integration ranges are studied. Theoretical derivations are performed and several case studies are developed to study the influences of correlation and truncation on plant revenue. Case studies show that, for a large enough CV of a product, assuming independent price and demand may underestimate the revenue by up to 20%. Since the real world demands or prices vary in limited ranges, integrating over the whole range of a normal distribution, which some research has done, may give incorrect results. This paper thus approximates a bivariate double-truncated normal distribution for demand and price. Case studies show that the degree of truncation can significantly influence the on plant revenue.

To handle possible unmet customer demands, the hard-to-specify penalty functions of the two-stage programming are avoided and replaced by two of the decision maker’s service level targets, namely the confidence level and fill rate target. Confidence level or the Type 1 service level is commonly used in chance-constrained programming. However, fill rate or the Type 2 service level is a greater concern of most managers. Two types of service levels, Type 1 and 2, are implemented into the planning model in this paper. Case studies show that a planning strategy that satisfies certain confidence level targets might be too generous compared to a strategy that satisfies a fill rate target. Case studies including refinery planning problems were used to illustrate the proposed approach.

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Appendix I Real world correlation coefficient estimation between demand and price

We estimate the correlation coefficients between demand and price for world crude oil and gasoline by regressing the real world 2003-2004 data from *EIA*[9].

The mean and standard deviation of the price and demand are estimated using:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{A1.1}$$

$$\bar{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} \tag{A1.2}$$

where, x_i is the sample data from *EIA* and *n* is the total number of sample data. $\bar{x}$ and $\bar{\sigma}$ are the UMVUE (Uniformly Minimum Variance Unbiased Estimators) of the mean and standard deviation.

The sample correlation coefficient is used as the estimator for ρ[10]:

$$\rho = \frac{\sum_{i=1}^{n} (c_i - \bar{c})(x_i - \bar{x})}{\sqrt{\sum_{i=1}^{n} (c_i - \bar{c})^2} \sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}} \tag{A1.3}$$

where, x_i and c_i are the sample demand and price data respectively. $\bar{x}$ and $\bar{c}$ are obtained using (A1.1).

The correlation coefficient between gasoline (New York Harbor Gasoline Regular) price and demand is 0.44 and 0.30 for world crude oil for year 2003 and 2004 using eq (A1.3) and the real world data from *EIA*. Figure A.1 shows the profile of the world crude oil demand and price (*EIA*). In Figure A.1, the crude oil price increases generally as the increase of crude oil demand.

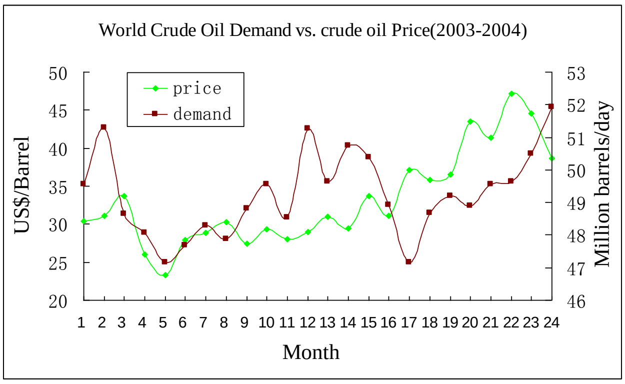

Figure A.1 World* crude oil demand vs. crude oil price**(2003-2004)

*: Refers to total OECD which includes OECD Europe, Canada, Japan, South Korea, United States, and Other.

**: Brent (US. Dollars per Barrel)

Appendix II The range of *x*

To increase accuracy, the polynomial approximation function, eq (17), is regressed in the range [–3, 3] and [–5, 5] because the domain of $\Phi(x)$ used in the single integrals locates in this range with high probability.

For a normal distribution with mean θ and standard deviation σ, we have

$$Pr\{\ \theta - 2\sigma < x < \theta + 2\sigma\} = 0.9545$$
$$Pr\{\ \theta - 3\sigma < x < \theta + 3\sigma\} = 0.9973$$

where Pr is the operator of the probability computation. In other words, there is a possibility of 99.73% that *x* locates in the range [θ–3σ, θ +3σ]. We have

$$\mathrm{U_U} = \frac{\frac{x_U - \theta}{\sigma_x} - \rho m}{\sqrt{1-\rho^2}} = \frac{\frac{x_U - \theta}{\sigma_x} - \rho \frac{c - \bar{c}}{\sigma_c}}{\sqrt{1-\rho^2}} \quad \text{and} \quad \mathrm{L} = \frac{\frac{x_L - \theta}{\sigma_x} - \rho m}{\sqrt{1-\rho^2}} = \frac{\frac{x_L - \theta}{\sigma_x} - \rho \frac{c - \bar{c}}{\sigma_c}}{\sqrt{1-\rho^2}}.$$

In the above equations, with a 99.73% confidence, when $x_U = \theta + 3\sigma_x$ and $c = \bar{c} - 3\sigma_c$, $\mathrm{U_U}$ is at its the upper bound, U_U^{UB}; when $x_L = \theta - 3\sigma_x$ and $c = \bar{c} + 3\sigma_c$, L is at its the lower bound, L^{LB}.

Thus, with a 99.73% confidence, we have the range of $\mathrm{U_U}$ and L:

$$U_U^{UB} = \frac{\dfrac{\theta + 3\sigma_x - \theta}{\sigma_x} - \rho\dfrac{\bar{c} - 3\sigma_c - \bar{c}}{\sigma_c}}{\sqrt{1-\rho^2}} = \frac{3+3\rho}{\sqrt{1-\rho^2}}$$

$$L^{LB} = \frac{\dfrac{\theta - 3\sigma_x - \theta}{\sigma_x} - \rho\dfrac{\bar{c} + 3\sigma_c - \bar{c}}{\sigma_c}}{\sqrt{1-\rho^2}} = \frac{-3-3\rho}{\sqrt{1-\rho^2}}$$

As *P* locates in the range [X_L, X_U], the range of U locates between U_U and L.

Similarly, we can obtain the ranges when the confidence is 95.45%. Table A.1 lists the ranges of L, U and U_U at different confidences and correlation coefficients. When the degree of truncation is $\mu \pm 2\sigma$, the ranges of L, U and U_U are listed in the first row of Table A.1 at different correlation coefficients. In this situation, we use [–3, 3] as the range of *x* and set a_0~a_n in eq (17) to values in the row [–3, 3] in Table 3. When the the degree of truncation is $\mu \pm \sigma$, the ranges of L, U and U_U are narrower than those in first row of Table A.1. Thus, we also can use [–3, 3] as the range of *x*. When the degree of truncation is $\mu \pm 3\sigma$, we use [–5, 5] as the range of *x* and set a_0~a_n in eq (17) to values in the row [–5, 5] in Table 3.

ρ	0	0.1	0.4	0.5
Range (95.45% confidence)	[–2.0, 2.0]	[–2.2, 2.2]	[–3.1, 3.1]	[–3.5, 3.5]
Range (99.73% confidence)	[–3.0, 3.0]	[–3.3, 3.3]	[–4.6, 4.6]	[–5.2, 5.2]

Table A.1 Range of *x*

Appendix III Some frequently used integrals for eq (9)

We derive the frequently used integrals in eq (9) (listed in Table 4) in this section.

$$\mathrm{E}_1 = \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} e^{-\frac{\mathrm{U}^2}{2}} dm = \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} e^{-\frac{(Wm+Z_U)^2}{2}} dm$$

$$= \int_{m=-\infty}^{+\infty} m e^{-\frac{[(1+W^2)(m+\frac{WZ_U}{1+W^2})^2 - \frac{W^2 Z_U^2}{1+W^2} + Z_U^2]}{2}} dm$$

$$= e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \int_{m=-\infty}^{+\infty} m e^{-\frac{(1+W^2)(m+\frac{WZ_U}{1+W^2})^2}{2}} dm$$

Let $t = \sqrt{1+W^2}\,(m + \frac{WZ_U}{1+W^2})$, then

$$\mathrm{E}_1 = e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{1+W^2}} (\frac{t}{\sqrt{1+W^2}} - \frac{WZ_U}{1+W^2}) e^{-\frac{t^2}{2}} dt$$

$$= e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} [\frac{1}{1+W^2} \int_{-\infty}^{+\infty} t e^{-\frac{t^2}{2}} dt - \frac{WZ_U}{(1+W^2)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} dt \qquad \text{(A3.1)}$$

$$= e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \sqrt{2\pi}\, \frac{-WZ_U}{(1+W^2)^{\frac{3}{2}}}$$

Similarly,

$$\mathrm{E}_3 = \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} e^{-\frac{\mathrm{U}^2}{2}} dm = e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \int_{m=-\infty}^{+\infty} e^{-\frac{(1+W^2)(m+\frac{WZ_U}{1+W^2})^2}{2}} dm = \frac{e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})}}{\sqrt{1+W^2}} \sqrt{2\pi} \qquad \text{(A3.2)}$$

$$M_0 = \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} dm = \sqrt{2\pi}\,, \quad M_1 = \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} dm = 0$$

$$M_2 = \int_{m=-\infty}^{+\infty} m^2 e^{-\frac{m^2}{2}} dm = -\int_{m=-\infty}^{+\infty} m\, d(e^{-\frac{m^2}{2}}) = -m e^{-\frac{m^2}{2}} |_{-\infty}^{+\infty} + \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} dm = M_0$$

$$M_3 = \int_{m=-\infty}^{+\infty} m^3 e^{-\frac{m^2}{2}} dm = 0\,, \quad M_4 = \int_{m=-\infty}^{+\infty} m^4 e^{-\frac{m^2}{2}} dm = 3\sqrt{2\pi}$$

In general,

$$M_{2n} = \int_{m=-\infty}^{+\infty} m^{2n} e^{-\frac{m^2}{2}} dm = \sqrt{2\pi} \prod_{i=1}^{n} (2n-2i+1)\,,\ n=1,2,3,\ldots \qquad \text{(A3.3)}$$

$$M_{2n-1} = \int_{m=-\infty}^{+\infty} m^{2n-1} e^{-\frac{m^2}{2}} dm = 0\,,\ n=1,2,3,\ldots \qquad \text{(A3.4)}$$

Appendix IV Some frequently used integrals for eq (16)

We derive the frequently used integrals in eq (16) (listed in Table 4) in this section.

Define $a = \frac{c_L - \bar{c}}{\sigma_c}$, $b = \frac{c_U - \bar{c}}{\sigma_c}$, we have

$$E_1^{'} = \int_{m=\frac{c_L-\bar{c}}{\sigma_c}}^{\frac{c_U-\bar{c}}{\sigma_c}} m e^{-\frac{m^2}{2}} e^{-\frac{U^2}{2}} dm = e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \int_a^b m e^{-\frac{(1+W^2)(m+\frac{WZ_U}{1+W^2})^2}{2}} dm$$

let $t = \sqrt{1+W^2}(m + \frac{WZ_U}{1+W^2})$, then

$$E_1^{'} = e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \int_{L_{E1}}^{U_{E1}} \frac{1}{\sqrt{1+W^2}} (\frac{t}{\sqrt{1+W^2}} - \frac{WZ_U}{1+W^2}) e^{-\frac{t^2}{2}} dt$$

$$= e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})} \left[\frac{-1}{1+W^2} (e^{-\frac{(U_{E1})^2}{2}} - e^{-\frac{(L_{E1})^2}{2}}) - \frac{WZ_U}{(1+W^2)^{\frac{3}{2}}} \sqrt{2\pi} [\Phi(U_{E1}) - \Phi(L_{E1})] \right] \quad \text{(A4.1)}$$

where $L_{E1} = a\sqrt{1+W^2} + \frac{WZ_U}{\sqrt{1+W^2}}$, $U_{E1} = b\sqrt{1+W^2} + \frac{WZ_U}{\sqrt{1+W^2}}$. Similarly,

$$E_2^{'} = \int_a^b m e^{-\frac{m^2}{2}} e^{-\frac{L^2}{2}} dm$$

$$= e^{-\frac{1}{2}(\frac{Z_L^2}{1+W^2})} \left[\frac{-1}{1+W^2} (e^{-\frac{(U_{E2})^2}{2}} - e^{-\frac{(L_{E2})^2}{2}}) - \frac{WZ_L}{(1+W^2)^{\frac{3}{2}}} \sqrt{2\pi} [\Phi(U_{E2}) - \Phi(L_{E2})] \right] \quad \text{(A4.2)}$$

where $L_{E2} = a\sqrt{1+W^2} + \frac{WZ_L}{\sqrt{1+W^2}}$, $U_{E2} = b\sqrt{1+W^2} + \frac{WZ_L}{\sqrt{1+W^2}}$.

$$E_3^{'} = \int_a^b e^{-\frac{m^2}{2}} e^{-\frac{U^2}{2}} dm = \frac{e^{-\frac{1}{2}(\frac{Z_U^2}{1+W^2})}}{\sqrt{1+W^2}} \sqrt{2\pi} [\Phi(U_{E1}) - \Phi(L_{E1})] \quad \text{(A4.3)}$$

$$E_4^{'} = \int_a^b e^{-\frac{m^2}{2}} e^{-\frac{L^2}{2}} dm = \frac{e^{-\frac{1}{2}(\frac{Z_L^2}{1+W^2})}}{\sqrt{1+W^2}} \sqrt{2\pi} [\Phi(U_{E2}) - \Phi(L_{E2})] \quad \text{(A4.4)}$$

$$M_0^{'} = \int_a^b e^{-\frac{m^2}{2}} dm = \sqrt{2\pi}[\Phi(b) - \Phi(a)], \quad M_1^{'} = \int_a^b m e^{-\frac{m^2}{2}} dm = -(e^{-\frac{b^2}{2}} - e^{-\frac{a^2}{2}})$$

In general,

$$M_n^{'} = \int_a^b m^n e^{-\frac{m^2}{2}} dm = -\int_a^b m^{n-1} d(e^{-\frac{m^2}{2}}) = -m^{n-1} e^{-\frac{m^2}{2}} |_a^b + (n-1)\int_a^b e^{-\frac{m^2}{2}} m^{n-2} dm$$
$$= -[b^{n-1} e^{-\frac{b^2}{2}} - a^{n-1} e^{-\frac{a^2}{2}}] + (n-1) M_{n-2}^{'}, \quad n = 2,3,4,... \qquad \text{(A4.5)}$$

Appendix V Properties of the revenue

In this appendix, we show that revenue formulae for correlated and truncated, eq (9) and eq (16), can be reduced to the formulae appeared in the literature when demand and price are independent and non-truncated. The normally distributed price and demand are independent if ρ=0. In this situation, we have

$$W = \frac{-\rho}{\sqrt{1-\rho^2}} = 0 ,$$

$\mathrm{U} = Z_U = \dfrac{P-\theta}{\sigma_x\sqrt{1-\rho^2}} = \dfrac{P-\theta}{\sigma_x}$, U becomes a constant.

From Table 2,

$$\mathrm{A1} = \frac{-\sqrt{1-\rho^2}\sigma_x\sigma_c}{2\pi} \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} e^{-\frac{\mathrm{U}^2}{2}} dm = \frac{-\sigma_x\sigma_c}{2\pi} e^{-\frac{\mathrm{U}^2}{2}} \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} dm = \frac{-\sigma_x\sigma_c}{2\pi} e^{-\frac{\mathrm{U}^2}{2}} M_1 = 0$$

$$\mathrm{A2} = \frac{-\sqrt{1-\rho^2}\sigma_x \bar{c}}{2\pi} \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} e^{-\frac{\mathrm{U}^2}{2}} dm = \frac{-\sigma_x \bar{c}}{2\pi} e^{-\frac{\mathrm{U}^2}{2}} \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} dm = \frac{-\sigma_x \bar{c}}{\sqrt{2\pi}} e^{-\frac{\mathrm{U}^2}{2}}$$

A3 =0,

$$\mathrm{A4} = \frac{\sigma_c\theta + \rho\bar{c}\sigma_x}{\sqrt{2\pi}} \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} \Phi(\mathrm{U}) dm = \frac{\sigma_c\theta}{\sqrt{2\pi}} \Phi(\mathrm{U}) \int_{m=-\infty}^{+\infty} m e^{-\frac{m^2}{2}} dm = 0$$

$$\mathrm{A5} = \frac{\bar{c}\theta}{\sqrt{2\pi}} \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} \Phi(\mathrm{U}) dm = \frac{\bar{c}\theta}{\sqrt{2\pi}} \Phi(\mathrm{U}) \int_{m=-\infty}^{+\infty} e^{-\frac{m^2}{2}} dm = \bar{c}\theta\Phi(\mathrm{U})$$

From eq (6),

$$\mathrm{C1} = P\bar{c} \int_{m=-\infty}^{+\infty} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} \Phi(\mathrm{U}) dm = P\bar{c}\Phi(\mathrm{U}) \int_{m=-\infty}^{+\infty} \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} dm = P\bar{c}\Phi(\mathrm{U})$$

$$C2 = P\sigma_c \int_{m=-\infty}^{+\infty} m \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} \Phi(\mathrm{U}) dm = P\sigma_c \Phi(\mathrm{U}) \int_{m=-\infty}^{+\infty} m \frac{e^{-\frac{m^2}{2}}}{\sqrt{2\pi}} dm = 0\,.$$

Thus, from eq (9),

$$\text{Revenue} = \text{A1} + \text{A2} + \text{A3} + \text{A4} + \text{A5} + \text{B} - \text{C1} - \text{C2} = \frac{-\sigma_x \bar{c}}{\sqrt{2\pi}} e^{-\frac{\mathrm{U}^2}{2}} + \bar{c}(\theta - P)\Phi(\mathrm{U}) + P\bar{c}$$

$$= -\sigma_x \bar{c}\phi(\frac{P-\theta}{\sigma_x}) + \bar{c}(\theta - P)\Phi(\frac{P-\theta}{\sigma_x}) + P\bar{c} = \bar{c}\theta + \sigma_x \bar{c}\left[-\phi(\frac{P-\theta}{\sigma_x}) + [1-\Phi(\frac{P-\theta}{\sigma_x})]\frac{P-\theta}{\sigma_x}\right]$$

where ϕ(.) is the standard normal density function, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}$. The above equation has reduced to the formulae appeared in the literature[4,6]. This also proves that, underlying the formulae used in the literature, the demand and price are assumed to be independent and take values in the range (–∞, +∞).

It is also easy to show that, when $x_U \to +\infty$ and $x_L \to -\infty$, $c_U \to +\infty$ and $c_L \to -\infty$, the equation for truncated and correlated revenue, eq (16), is reduced to the equation for non-truncated and correlated revenue, eq (9). In fact, when $x_L \to -\infty$ then $e^{-\frac{\mathrm{L}^2}{2}} \to 0$, $\Phi(\mathrm{L}) \to 0$; when $x_U \to +\infty$ then $\mathrm{U_U} \to +\infty$, $\Phi(\mathrm{U_U}) \to 1$. We also have $F_{LU} \to 1$. Thus, the single integrals for $\mathrm{A_T}$ tend to the single integrals for A (Table 2), i.e., $\mathrm{A1_T \sim A5_T} \to \mathrm{A1 \sim A5}$, respectively. Furthermore, $\mathrm{C1_T} \to P\bar{c} - \mathrm{C1} = \mathrm{B} - \mathrm{C1}$; $\mathrm{C2_T} \to -\mathrm{C2}$. Hence, eq (16) reduces to eq (9).

Appendix VI Derivation of the bi-truncated expectation and loss function

The expectation of the bi-truncated normal demand, θ_{BTN}, is:

$$\theta_{BTN} = \int_{x_L}^{x_U} x\rho_{BTN}(x)dx = \frac{1}{\sigma_{BTN}} \int_{x_L}^{x_U} x\phi(\frac{x-\theta}{\sigma_x})dx$$

let $t = \dfrac{x-\theta}{\sigma_x}$, then

$$\theta_{BTN} = \frac{1}{\sigma_{BTN}} \int_{Z_{XL}}^{Z_{XU}} (t\sigma_x + \theta) \frac{\sigma_x}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt = \frac{\sigma_x^2}{\sigma_{BTN}\sqrt{2\pi}} \int_{Z_{XL}}^{Z_{XU}} te^{\frac{-t^2}{2}} dt + \frac{\theta\sigma_x}{\sigma_{BTN}\sqrt{2\pi}} \int_{Z_{XL}}^{Z_{XU}} e^{\frac{-t^2}{2}} dt$$

$$= \frac{-\sigma_x^2}{\sigma_{BTN}} [\phi(Z_{XU}) - \phi(Z_{XL})] + \frac{\theta\sigma_x}{\sigma_{BTN}} [\Phi(Z_{XU}) - \Phi(Z_{XL})]$$

Since $\sigma_{BTN} = \sigma_x[\Phi(Z_{XU}) - \Phi(Z_{XL})]$, we have

$$\theta_{BTN} = \theta - \sigma_x \frac{\phi(Z_{XU}) - \phi(Z_{XL})}{\Phi(Z_{XU}) - \Phi(Z_{XL})} \tag{A5.1}$$

The bi-truncated loss function, $LF_{BTN}(P)$, is:

$$LF_{BTN}(P) = \int_P^{X_U} (x-P)\rho_{BTN}(x)dx = \frac{1}{\sigma_{BTN}} \int_P^{X_U} x\phi(\frac{x-\theta}{\sigma_x})dx - \frac{P}{\sigma_{BTN}} \int_P^{X_U} \phi(\frac{x-\theta}{\sigma_x})dx$$

let $t = \dfrac{x-\theta}{\sigma_x}$ and define $Z_{XP} = \dfrac{P-\theta}{\sigma_x}$, then

$$LF_{BTN}(P) = \frac{1}{\sigma_{BTN}} \int_{Z_{XP}}^{Z_{XU}} (t\sigma_x + \theta) \frac{\sigma_x}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt - \frac{P}{\sigma_{BTN}} \int_{Z_{XP}}^{Z_{XU}} \frac{\sigma_x}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$$

$$= \frac{\sigma_X^2}{\sigma_{BTN}\sqrt{2\pi}} \int_{Z_{XP}}^{Z_{XU}} te^{\frac{-t^2}{2}} dt + \frac{\theta\sigma_x}{\sigma_{BTN}\sqrt{2\pi}} \int_{Z_{XP}}^{Z_{XU}} e^{\frac{-t^2}{2}} dt - \frac{P\sigma_x}{\sigma_{BTN}\sqrt{2\pi}} \int_{Z_{XP}}^{Z_{XU}} e^{\frac{-t^2}{2}} dt$$

Thus

$$LF_{BTN}(P) = \sigma_x\{\frac{-\sigma_x}{\sigma_{BTN}}[\phi(Z_{XU}) - \phi(Z_{XP})] - \frac{(P-\theta)}{\sigma_{BTN}}[\Phi(Z_{XU}) - \Phi(Z_{XP})]\} \tag{A5.2}$$

When the demand conforms to left truncated normal distribution**,** that is, $X_U \to +\infty$ and $Z_{XU} \to +\infty$, we have

$\Phi(Z_{XU}) \to 1,\ \phi(Z_{XU}) \to 0$ and $\sigma_{BTN} \to \sigma_{LTN} = \sigma_x[1 - \Phi(Z_{XL})]$. Thus the left truncated loss function, $LF_{LTN}(P)$,

is

$$LF_{LTN}(P) = \sigma_x\{\frac{\sigma_x\phi(Z_{XP})}{\sigma_{LTN}} - \frac{(P-\theta)}{\sigma_{LTN}}[1 - \Phi(Z_{XP})]\} \tag{A5.3}$$

When the demand conforms to right truncated normal distribution**,** i.e., $X_L \to -\infty$ and $Z_{XL} \to -\infty$, then

$\Phi(Z_{XL}) \to 0,\ \phi(Z_{XL}) \to 0$ and $\sigma_{BTN} \to \sigma_{RTN} = \sigma_x\Phi(Z_{XU})$. Thus the right truncated loss function, $LF_{RTN}(P)$, is

$$LF_{RTN}(P) = \frac{-\sigma_x}{\Phi(Z_{XU})}[\phi(Z_{XU}) - \phi(Z_{XP})] - \frac{(P-\theta)}{\Phi(Z_{XU})}[\Phi(Z_{XU}) - \Phi(Z_{XP})] \tag{A5.4}$$

For non-truncated normal distribution, i.e., $X_U \to +\infty$ and $X_L \to -\infty$, we have $Z_{XU} \to +\infty$ and $Z_{XL} \to -\infty$

and $\sigma_{BTN} \to \sigma_x$. Eq (A5.2) is reduced to

$$LF(P) = \sigma_x \{\phi(Z_{XP}) - Z_{XP}[1 - \Phi(Z_{XP})]\} \tag{A5.5}$$

The above formula can also be found in the literature[4,6].